Lecture 6
Invariant subspaces

- invariant subspaces
- a matrix criterion
- Sylvester equation
- the PBH controllability and observability conditions
- invariant subspaces, quadratic matrix equations, and the ARE

Invariant subspaces

suppose $A \in \mathbf{R}^{n\times n}$ and $\mathcal{V} \subseteq \mathbf{R}^n$ is a subspace

we say that $\mathcal{V}$ is A-*invariant* if $A\mathcal{V} \subseteq \mathcal{V}$, $i.e.$, $v \in \mathcal{V} \implies Av \in \mathcal{V}$

examples:

- $\{0\}$ and $\mathbf{R}^n$ are always A-invariant
- $\mathrm{span}\{v_1, \ldots, v_m\}$ is A-invariant, where v_i are (right) eigenvectors of A
- if A is block upper triangular,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

with $A_{11} \in \mathbf{R}^{r\times r}$, then $\mathcal{V} = \left\{ \begin{bmatrix} z \\ 0 \end{bmatrix} \;\middle|\; z \in \mathbf{R}^r \right\}$ is A-invariant

Examples from linear systems

- if $B \in \mathbf{R}^{n \times m}$, then the controllable subspace

$$\mathcal{R}(\mathcal{C}) = \mathcal{R}\left([B \ AB \ \cdots \ A^{n-1}B]\right)$$

is A-invariant

- if $C \in \mathbf{R}^{p \times n}$, then the unobservable subspace

$$\mathcal{N}(\mathcal{O}) = \mathcal{N}\left(\left[\begin{array}{c} C \\ \vdots \\ CA^{n-1} \end{array}\right]\right)$$

is A-invariant

Dynamical interpretation

consider system $\dot{x} = Ax$

$\mathcal{V}$ is A-invariant if and only if

$$x(0) \in \mathcal{V} \implies x(t) \in \mathcal{V} \text{ for all } t \geq 0$$

(same statement holds for discrete-time system)

A matrix criterion for A-invariance

suppose $\mathcal{V}$ is A-invariant

let columns of $M \in \mathbf{R}^{n \times k}$ span $\mathcal{V}$, *i.e.*,

$$\mathcal{V} = \mathcal{R}(M) = \mathcal{R}([t_1 \ \cdots \ t_k])$$

since $At_1 \in \mathcal{V}$, we can express it as

$$At_1 = x_{11}t_1 + \cdots + x_{k1}t_k$$

we can do the same for $At_2, \ldots, At_k$, which gives

$$A[t_1 \ \cdots \ t_k] = [t_1 \ \cdots \ t_k] \begin{bmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} \end{bmatrix}$$

or, simply, $AM = MX$

in other words: if $\mathcal{R}(M)$ is A-invariant, then there is a matrix X such that $AM = MX$

converse is also true: if there is an X such that $AM = MX$, then $\mathcal{R}(M)$ is A-invariant

now assume M is rank k, *i.e.*, $\{t_1, \ldots, t_k\}$ is a basis for $\mathcal{V}$

then every eigenvalue of X is an eigenvalue of A, and the associated eigenvector is in $\mathcal{V} = \mathcal{R}(M)$

if $Xu = \lambda u$, $u \neq 0$, then $Mu \neq 0$ and $A(Mu) = MXu = \lambda Mu$

so the eigenvalues of X are a subset of the eigenvalues of A

more generally: if $AM = MX$ (no assumption on rank of M), then A and X share at least $\mathbf{Rank}(M)$ eigenvalues

Sylvester equation

the *Sylvester equation* is $AX + XB = C$, where $A,\ B,\ C,\ X \in \mathbf{R}^{n\times n}$

when does this have a solution X for every C?

express as $S(X) = C$, where S is the linear function $S(X) = AX + XB$ (S maps $\mathbf{R}^{n\times n}$ into $\mathbf{R}^{n\times n}$ and is called the *Sylvester operator*)

so the question is: when is S nonsingular?

S is singular if and only if there exists a nonzero X with $S(X) = 0$

this means $AX + XB = 0$, so $AX = X(-B)$, which means A and $-B$ share at least one eigenvalue (since $X \neq 0$)

so we have: if S is singular, then A and $-B$ have a common eigenvalue let's show the converse: if A and $-B$ share an eigenvalue, S is singular

suppose

$$Av = \lambda v, \qquad w^T B = -\lambda w^T, \qquad v,\ w \neq 0$$

then with $X = vw^T$ we have $X \neq 0$ and

$$S(X) = AX + XB = Avw^T + vw^T B = (\lambda v)w^T + v(-\lambda w^T) = 0$$

which shows S is singular

so, Sylvestor operator is singular if and only if A and $-B$ have a common eigenvalue

or: Sylvestor operator is nonsingular if and only if A and $-B$ have no common eigenvalues

Uniqueness of stabilizing ARE solution

suppose P is any solution of ARE

$$A^TP + PA + Q - PBR^{-1}B^TP = 0$$

and define $K = -R^{-1}B^TP$

we say P is a *stabilizing solution* of ARE if

$$A + BK = A - BR^{-1}B^TP$$

is stable, $i.e.$, its eigenvalues have negative real part

fact: there is at most one stabilizing solution of the ARE
(which therefore is the one that gives the value function)

to show this, suppose P_1 and P_2 are both stabilizing solutions

subtract AREs to get

$$A^T(P_1 - P_2) + (P_1 - P_2)A - P_1BR^{-1}B^TP_1 + P_2BR^{-1}B^TP_2 = 0$$

rewrite as Sylvester equation

$$(A + BK_2)^T(P_1 - P_2) + (P_1 - P_2)(A + BK_1) = 0$$

since $A + BK_2$ and $A + BK_1$ are both stable, $A + BK_2$ and $-(A + BK_1)$ cannot share any eigenvalues, so we conclude $P_1 - P_2 = 0$

Change of coordinates

suppose $\mathcal{V} = \mathcal{R}(M)$ is A-invariant, where $M \in \mathbf{R}^{n\times k}$ is rank k

find $\tilde{M} \in \mathbf{R}^{n\times(n-k)}$ so that $[M\ \tilde{M}]$ is nonsingular

$$A[M\ \tilde{M}] = [AM\ A\tilde{M}] = [M\ \tilde{M}] \left[\begin{array}{cc} X & Y \\ 0 & Z \end{array}\right]$$

where

$$\left[\begin{array}{c} Y \\ Z \end{array}\right] = [M\ \tilde{M}]^{-1} A\tilde{M}$$

with $T = [M\ \tilde{M}]$, we have

$$T^{-1}AT = \left[\begin{array}{cc} X & Y \\ 0 & Z \end{array}\right]$$

in other words: if $\mathcal{V}$ is A-invariant we can change coordinates so that

- A becomes block upper triangular in the new coordinates
- $\mathcal{V}$ corresponds to $\left\{ \left[\begin{array}{c} z \\ 0 \end{array} \right] \;\middle|\; z \in \mathbf{R}^k \right\}$ in the new coordinates

Revealing the controllable subspace

consider $\dot{x} = Ax + Bu$ (or $x_{t+1} = Ax_t + Bu_t$) and assume it is *not* controllable, so $\mathcal{V} = \mathcal{R}(\mathcal{C}) \neq \mathbf{R}^n$

let columns of $M \in \mathbf{R}^k$ be basis for controllable subspace ($e.g.$, choose k independent columns from $\mathcal{C}$)

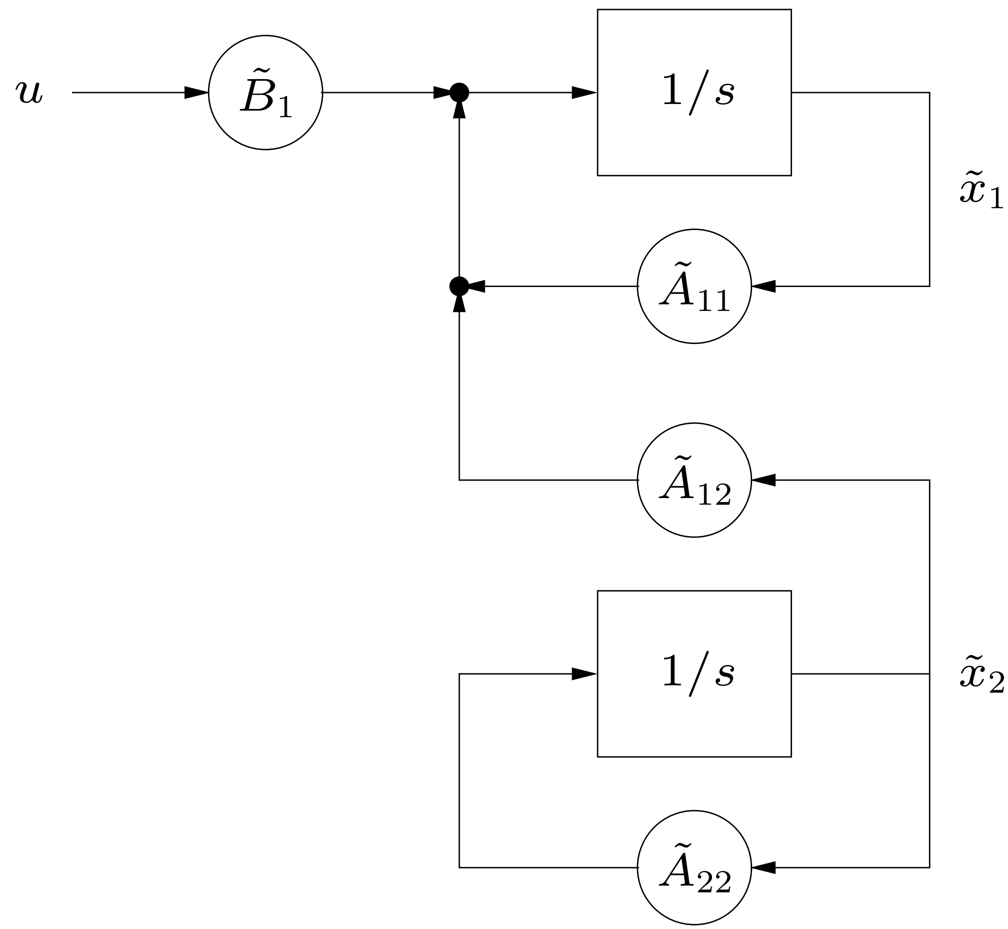
let $\tilde{M} \in \mathbf{R}^{n\times(n-k)}$ be such that $T = [M\ \tilde{M}]$ is nonsingular

then

$$T^{-1}AT = \left[\begin{array}{cc} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{array}\right], \qquad T^{-1}B = \left[\begin{array}{c} \tilde{B}_1 \\ 0 \end{array}\right]$$

$$\tilde{\mathcal{C}} = T^{-1}\mathcal{C} = \left[\begin{array}{ccc} \tilde{B}_1 & \cdots & \tilde{A}_{11}^{n-1}\tilde{B}_1 \\ 0 & \cdots & 0 \end{array}\right]$$

in the new coordinates the controllable subspace is $\{(z,0) \mid z \in \mathbf{R}^k\}$; $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable we have changed coordinates to reveal the controllable subspace:



roughly speaking, $\tilde{x}_1$ is the controllable part of the state

Revealing the unobservable subspace

similarly, if (C, A) is not observable, we can change coordinates to obtain

$$T^{-1}AT = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \qquad CT = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix}$$

and $(\tilde{C}_1, \tilde{A}_{11})$ is observable

Popov-Belevitch-Hautus controllability test

PBH controllability criterion: (A, B) is controllable if and only if

$$\mathbf{Rank}\ [sI - A\ B] = n \text{ for all } s \in \mathbf{C}$$

equivalent to:

(A, B) is uncontrollable if and only if there is a $w \neq 0$ with

$$w^T A = \lambda w^T, \qquad w^T B = 0$$

$i.e.$, a left eigenvector is orthogonal to columns of B

to show it, first assume that $w \neq 0$, $w^T A = \lambda w^T$, $w^T B = 0$

then for $k = 1, \dots, n-1$, $w^T A^k B = \lambda^k w^T B = 0$, so

$$w^T [B \ AB \ \cdots \ A^{n-1}B] = w^T \mathcal{C} = 0$$

which shows (A, B) not controllable

conversely, suppose (A, B) not controllable

change coordinates as on p.6–15, let z be any left eigenvector of $\tilde{A}_{22}$, and define $\tilde{w} = (0, z)$

then $\tilde{w}^T \tilde{A} = \lambda \tilde{w}^T$, $\tilde{w}^T \tilde{B} = 0$

it follows that $w^T A = \lambda w^T$, $w^T B = 0$, where $w = T^{-T} \tilde{w}$

PBH observability test

PBH observability criterion: (C, A) is observable if and only if

$$\mathbf{Rank} \begin{bmatrix} sI - A \\ C \end{bmatrix} = n \text{ for all } s \in \mathbf{C}$$

equivalent to:

(C, A) is unobservable if and only if there is a $v \neq 0$ with

$$Av = \lambda v, \qquad Cv = 0$$

$i.e.$, a (right) eigenvector is in the nullspace of C

Observability and controllability of modes

the PBH tests allow us to identify unobservable and uncontrollable modes

the mode associated with right and left eigenvectors $v,\ w$ is

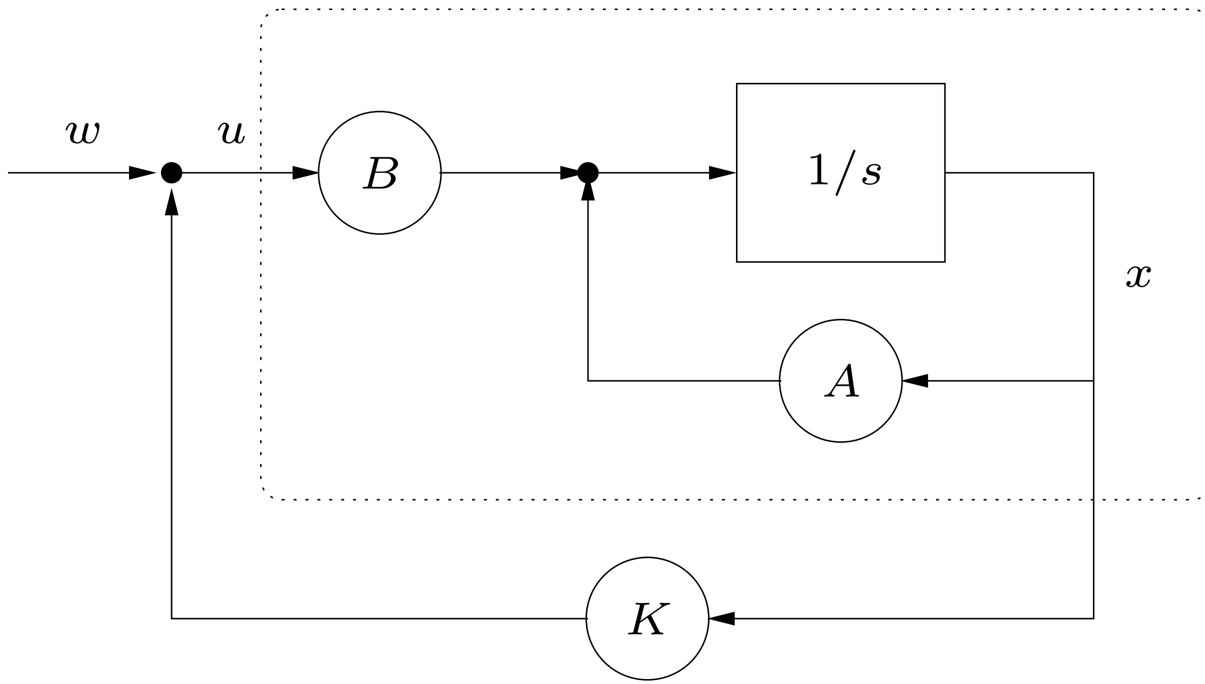
- uncontrollable if $w^T B = 0$
- unobservable if $Cv = 0$

(classification can be done with repeated eigenvalues, Jordan blocks, but gets tricky)

Controllability and linear state feedback

we consider system $\dot{x} = Ax + Bu$ (or $x_{t+1} = Ax_t + Bu_t$)

we refer to $u = Kx + w$ as a *linear state feedback* (with auxiliary input w), with associated *closed-loop system* $\dot{x} = (A + BK)x + Bw$



suppose $w^T A = \lambda w^T$, $w \neq 0$, $w^T B = 0$, $i.e.$, w corresponds to uncontrollable mode of open loop system

then $w^T(A + BK) = w^T A + w^T BK = \lambda w^T$, $i.e.$, w is also a left eigenvector of closed-loop system, associated with eigenvalue λ

$i.e.$, eigenvalues (and indeed, left eigenvectors) associated with uncontrollable modes cannot be changed by linear state feedback

conversely, if w is left eigenvector associated with uncontrollable closed-loop mode, then w is left eigenvector associated with uncontrollable open-loop mode

in other words: state feedback preserves uncontrollable eigenvalues and the associated left eigenvectors

Invariant subspaces and quadratic matrix equations

suppose $\mathcal{V} = \mathcal{R}(M)$ is A-invariant, where $M \in \mathbf{R}^{n\times k}$ is rank k, so $AM = MX$ for some $X \in \mathbf{R}^{k\times k}$

conformally partition as

$$\left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right]\left[\begin{array}{c} M_1 \\ M_2 \end{array}\right] = \left[\begin{array}{c} M_1 \\ M_2 \end{array}\right] X$$

$$A_{11}M_1 + A_{12}M_2 = M_1X, \qquad A_{21}M_1 + A_{22}M_2 = M_2X$$

eliminate X from first equation (assuming M_1 is nonsingular):

$$X = M_1^{-1}A_{11}M_1 + M_1^{-1}A_{12}M_2$$

substituting this into second equation yields

$$A_{21}M_1 + A_{22}M_2 = M_2M_1^{-1}A_{11}M_1 + M_2M_1^{-1}A_{12}M_2$$

multiply on right by M_1^{-1}:

$$A_{21} + A_{22}M_2M_1^{-1} = M_2M_1^{-1}A_{11} + M_2M_1^{-1}A_{12}M_2M_1^{-1}$$

with $P = M_2M_1^{-1}$, we have

$$-A_{22}P + PA_{11} - A_{21} + PA_{12}P = 0,$$

a general quadratic matrix equation

if we take A to be Hamitonian associated with a cts-time LQR problem, we recover the method of solving ARE via stable eigenvectors of Hamiltonian