Lecture 14
Lyapunov theory with inputs and outputs

- systems with inputs and outputs
- reachability bounding
- bounds on RMS gain
- bounded-real lemma
- feedback synthesis via control-Lyapunov functions

Systems with inputs

we now consider systems with inputs, *i.e.*, $\dot{x} = f(x,u)$, where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$

if $x,\ u$ is state-input trajectory and $V : \mathbf{R}^n \to \mathbf{R}$, then

$$\frac{d}{dt}V(x(t)) = \nabla V(x(t))^T \dot{x}(t) = \nabla V(x(t))^T f(x(t), u(t))$$

so we define $\dot{V} : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ as

$$\dot{V}(z,w) = \nabla V(z)^T f(z,w)$$

(*i.e.*, $\dot{V}$ depends on the state and input)

Reachable set with admissible inputs

consider $\dot{x} = f(x, u)$, $x(0) = 0$, and $u(t) \in \mathcal{U}$ for all t

$\mathcal{U} \subseteq \mathbf{R}^m$ is called the set of *admissable inputs*

we define the *reachable set* as

$$\mathcal{R} = \{x(T) \mid \dot{x} = f(x, u),\ x(0) = 0,\ u(t) \in \mathcal{U},\ T > 0\}$$

$i.e.$, the set of points that can be hit by a trajectory with some admissiable input

applications:

- if u is a control input that we can manipulate, $\mathcal{R}$ shows the places we can hit (so big $\mathcal{R}$ is good)
- if u is a disturbance, noise, or antagonistic signal (beyond our control), $\mathcal{R}$ shows the worst-case effect on x (so big $\mathcal{R}$ is bad)

Lyapunov bound on reachable set

Lyapunov arguments can be used to bound reachable sets of nonlinear or time-varying systems

suppose there is a $V : \mathbf{R}^n \to \mathbf{R}$ and $a > 0$ such that

$$\dot{V}(z, w) \leq -a \text{ whenever } V(z) = b \text{ and } w \in \mathcal{U}$$

and define $C = \{z \mid V(z) \leq b\}$

then, if $\dot{x} = f(x, u)$, $x(0) \in C$, and $u(t) \in \mathcal{U}$ for $0 \leq t \leq T$, we have $x(T) \in C$

$i.e.$, every trajectory that starts in $C = \{z \mid V(z) \leq b\}$ stays there, for any admissable u

in particular, if $0 \in C$, we conclude $\mathcal{R} \subseteq C$

idea: on the boundary of C, every trajectory cuts *into* C, for all admissable values of u

proof: suppose $\dot{x} = f(x, u)$, $x(0) \in C$, and $u(t) \in \mathcal{U}$ for $0 \leq t \leq T$, and V satisfies hypotheses

suppose that $x(T) \notin C$

consider scalar function $g(t) = V(x(t))$

$g(0) \leq b$ and $g(T) > b$, so there is a $t_0 \in [0, T]$ with $g(t_0) = b$, $g'(t_0) \geq 0$

but

$$g'(t_0) = \frac{d}{dt} V(x(t)) = \dot{V}(x(t), u(t)) \leq -a < 0$$

by the hypothesis, so we have a contradiction

Reachable set with integral quadratic bounds

we consider $\dot{x} = f(x,u)$, $x(0) = 0$, with an integral constraint on the input:

$$\int_0^\infty u(t)^T u(t)\, dt \le a$$

the reachable set with this integral quadratic bound is

$$\mathcal{R}_a = \left\{ x(T) \;\middle|\; \dot{x} = f(x,u),\ x(0) = x_0,\ \int_0^T u(t)^T u(t)\, dt \le a \right\}$$

$i.e.$, the set of points that can be hit using at most a energy

Example

consider stable linear system $\dot{x} = Ax + Bu$

minimum energy (*i.e.*, integral of $u^T u$) to hit point z is $z^T W_c^{-1} z$, where W_c is controllability Grammian

reachable set with integral quadratic bound is (open) ellipsoid

$$\mathcal{R}_a = \{z \mid z^T W_c^{-1} z < a\}$$

Lyapunov bound on reachable set with integral constraint

suppose there is a $V : \mathbf{R}^n \to \mathbf{R}$ such that

- $V(z) \geq 0$ for all z, $V(0) = 0$
- $\dot{V}(z, w) \leq w^T w$ for all $z,\ w$

then $\mathcal{R}_a \subseteq \{z \mid V(z) \leq a\}$

proof:

$$V(x(T)) - V(x(0)) = \int_0^T \dot{V}(x(t), u(t))\ dt \leq \int_0^T u(t)^T u(t)\ dt \leq a$$

so, using $V(x(0)) = V(0) = 0$, $V(x(T)) \leq a$

interpretation:

- V is (generalized) internally stored energy in system
- $u(t)^T u(t)$ is power supplied to system by input
- $\dot{V} \leq u^T u$ means stored energy increases by no more than power input
- $V(0) = 0$ means system starts in zero energy state
- conclusion is: if energy $\leq a$ applied, can only get to states with stored energy $\leq a$

Stable linear system

consider stable linear system $\dot{x} = Ax + Bu$

we'll show Lyapunov bound is tight in this case, with $V(z) = z^T W_c^{-1} z$

multiply $AW_c + W_c A^T + BB^T = 0$ on left & right by W_c^{-1} to get

$$W_c^{-1}A + A^T W_c^{-1} + W_c^{-1}BB^T W_c^{-1} = 0$$

now we can find and bound $\dot{V}$:

$$\begin{aligned}
\dot{V}(z,w) &= 2z^T W_c^{-1}(Az + Bw) \\
&= z^T \left(W_c^{-1}A + A^T W_c^{-1}\right) z + 2z^T W_c^{-1} Bw \\
&= -z^T W_c^{-1} BB^T W_c^{-1} z + 2z^T W_c^{-1} Bw \\
&= -\|B^T W_c^{-1} z - w\|^2 + w^T w \\
&\leq w^T w
\end{aligned}$$

for $V(z) = z^T W_c^{-1} z$, Lyapunov bound is

$$\mathcal{R}_a \subseteq \{z \mid z^T W_c^{-1} z \leq a\}$$

righthand set is closure of lefthand set, so bound is tight

roughly speaking, for a stable linear system, a point is reachable with an integral quadratic bound if and only if there is a quadratic Lyapunov function that proves it
(except for points right on the boundary)

RMS gain

recall that the RMS value of a signal is given by

$$\textbf{rms}(z) = \left(\lim_{T\to\infty} \frac{1}{T} \int_0^T \|z(t)\|^2 \, dt \right)^{1/2}$$

assuming the limit exists

now consider a system with input signal u and output signal y

we define its *RMS gain* as the maximum of $\textbf{rms}(y)/\textbf{rms}(u)$, over all u with nonzero RMS value

Lyapunov method for bounding RMS gain

now consider the nonlinear system

$$\dot{x} = f(x,u), \qquad x(0) = 0, \qquad y = g(x,u)$$

with $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, $y(t) \in \mathbf{R}^p$

we can use Lyapunov methods to bound its RMS gain

suppose $\gamma \geq 0$, and there is a $V : \mathbf{R}^n \to \mathbf{R}$ such that

- $V(z) \geq 0$ for all z, $V(0) = 0$
- $\dot{V}(z,w) \leq \gamma^2 w^T w - y^T y$ for all $z,\ w$
  ($i.e.$, $\dot{V}(z,w) \leq \gamma^2 w^T w - g(z,w)^T g(z,w)$ for all $z,\ w$)

then, the RMS gain of the system is no more than γ

proof:

$$\begin{aligned} V(x(T)) - V(x(0)) &= \int_0^T \dot{V}(x(t), u(t))\, dt \\ &\leq \int_0^T \left(\gamma^2 u(t)^T u(t) - y(t)^T y(t)\right)\, dt \end{aligned}$$

using $V(x(0)) = V(0) = 0$, $V(x(T)) \geq 0$, we have

$$\int_0^T y(t)^T y(t)\, dt \leq \gamma^2 \int_0^T u(t)^T u(t)\, dt$$

dividing by T and taking the limit $T \to \infty$ yields $\mathbf{rms}(y)^2 \leq \gamma^2 \mathbf{rms}(u)^2$

Bounded-real lemma

let's use a quadratic Lyapunov function $V(z) = z^T Pz$ to bound the RMS gain of the stable linear system $\dot{x} = Ax + Bu$, $x(0) = 0$, $y = Cx$

the conditions on V give $P \geq 0$

the condition $\dot{V}(z, w) \leq \gamma^2 w^T w - g(z, w)^T g(z, w)$ becomes

$$\dot{V}(z, w) = 2z^T P(Az + Bw) \leq \gamma^2 w^T w - (Cz)^T Cz$$

for all $z,\ w$

let's write that as a quadratic form in (z, w):

$$\left[\begin{array}{c} z \\ w \end{array}\right]^T \left[\begin{array}{cc} A^T P + PA + C^T C & PB \\ B^T P & -\gamma^2 I \end{array}\right] \left[\begin{array}{c} z \\ w \end{array}\right] \leq 0$$

so we conclude: if there is a $P \geq 0$ such that

$$\begin{bmatrix} A^T P + PA + C^T C & PB \\ B^T P & -\gamma^2 I \end{bmatrix} \leq 0$$

then the RMS gain of the linear system is no more than γ

it turns out that for linear systems this condition is not only sufficient, but also necessary

(this result is called the *bounded-real lemma*)

by taking Schur complement, we can express the block 2×2 matrix inequality as

$$A^T P + PA + C^T C + \gamma^{-2} PBB^T P \leq 0$$

(which is a Riccati-like quadratic matrix *inequality* . . .)

Nonlinear optimal control

we consider $\dot{x} = f(x, u)$, $u(t) \in \mathcal{U} \subseteq \mathbf{R}^m$

here we consider u to be an input we can manipulate to achieve some desired response, such as minimizing, or at least making small,

$$J = \int_0^\infty x(t)^T Q x(t)\, dt$$

where $Q \geq 0$

(many other choices for criterion will work) we can solve via dynamic programming: let $V : \mathbf{R}^n \to \mathbf{R}$ denote value function, *i.e.*,

$$V(z) = \min\{J \mid \dot{x} = f(x,u),\ x(0) = z,\ u(t) \in \mathcal{U}\}$$

then the optimal u is given by

$$u^*(t) = \operatorname*{argmin}_{w \in \mathcal{U}} \dot{V}(x(t), w)$$

and with the optimal u we have

$$\dot{V}(x(t), u^*) = -x(t)^T Q x(t)$$

but, it can be very difficult to find V, and therefore u^*

Feedback design via control-Lyapunov functions

suppose there is a function $V : \mathbf{R}^n \to \mathbf{R}$ such that

- $V(z) \geq 0$ for all z
- for all z, $\min_{w \in \mathcal{U}} \dot{V}(z, w) \leq -z^T Q z$

then, the state feedback control law $u(t) = g(x(t))$, with

$$g(z) = \operatorname*{argmin}_{w \in \mathcal{U}} \dot{V}(z, w)$$

results in $J \leq V(x(0))$

in this case V is called a *control-Lyapunov* function for the problem

- if V is the value function, this method recovers the optimal control law

- we've used Lyapunov methods to generate a suboptimal control law, but one with a guaranteed bound on the cost function

- the control law is a greedy one, that simply chooses $u(t)$ to decrease V as quickly as possible (subject to $u(t) \in \mathcal{U}$)

- the inequality $\min_{w \in \mathcal{U}} \dot{V}(z, w) \leq -z^T Q z$ is the inequality form of $\min_{w \in \mathcal{U}} \dot{V}(z, w) = -z^T Q z$, which holds for the optimal input, and V the value function

control-Lyapunov methods offer a good way to generate suboptimal control laws, with performance guarantees, when the optimal control is too hard to find