# Locally positive definite function

function $V : \mathbf{R}^n \rightarrow \mathbf{R}$ is *locally positive definite* (LPD) if there exists an $R > 0$ such that

- $V(0) = 0$
- $V(z) > 0$ for all $\|z\| \leq R$, $z \neq 0$

## A Lyapunov local asymptotic stability theorem

suppose $x_e = 0$ is an equilibrium point of $\dot{x} = f(x)$

suppose there is a function $V$ such that

- there exists $R > 0$ such that $V$ is LPD on $B_R$
- $\dot{V}(z) < 0$ for all $z \neq 0$ in $B_R$
- there exists $\alpha > 0$ such that $C = \{z \mid V(z) \leq \alpha\} \subset B_R$

then, the system is locally asymptotically stable about $0$, $i.e.$, there exists $r > 0$ such that for all trajectories $x(t)$ with $\|x(0)\| \leq r$, $x(t) \to 0$ as $t \to \infty$

# Proof

Choose $r > 0$ such that $B_r \subset C$.

Suppose there is a trajectory $x(t)$, with $x(0) \in B_r$, that does not converge to $0$

$\dot{V} \leq 0$ for $z \in C \Rightarrow V(x(t)) \leq V(x(0))$, so $C$ is invariant

$V(x(t))$ is decreasing and bounded below by $0$ ($V(x(t)) \in C$ for all $t$), so it must converge to some limit $\epsilon > 0$

$\dot{V} \leq -a < 0$ on $\{z \mid \epsilon \leq V(z) \leq V(x(0))\}$, since $\dot{V}$ attains its supremum on this compact set

so $V(x(T)) \leq V(x(0)) - aT \Rightarrow V(x(T)) < 0$ for $T > V(x(0))/a$, which is a contradiction since $V \geq 0$ on $C$

so every trajectory with $\|x(0)\| \leq r$ converges to $0$, $i.e.$, system is locally asymptotically stable