Robust Optimization

- definitions of robust optimization
- robust linear programs
- robust cone programs
- chance constraints
- distributional robustness

Robust optimization

convex objective $f_0 : \mathbf{R}^n \to \mathbf{R}$, *uncertainty set* $\mathcal{U}$, and $f_i : \mathbf{R}^n \times \mathcal{U} \to \mathbf{R}$,

$$x \mapsto f_i(x, u) \text{ convex for all } u \in \mathcal{U}$$

general form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x,u) \leq 0 \text{ for all } u \in \mathcal{U}, i = 1, \dots, m. \end{array}$$

equivalent to

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & \sup_{u \in \mathcal{U}} f_i(x,u) \leq 0, i = 1, \dots, m. \end{array}$$

- Bertsimas, Ben-Tal, El-Ghaoui, Nemirovski (1990s–now)

Setting up robust problem

- can always replace objective f_0 with $\sup_{u \in \mathcal{U}} f_0(x, u)$, rewrite in epigraph form to

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \sup_u f_0(x,u) \le t, \sup_u f_i(x,u) \le 0, \;\; i = 1, \ldots, m \end{array}$$

- equality constraints make no sense: a robust equality $a^T(x + u) = b$ for all $u \in \mathcal{U}$?

three questions:

- is robust formulation useful?
- is robust formulation computable?
- how should we choose $\mathcal{U}$?

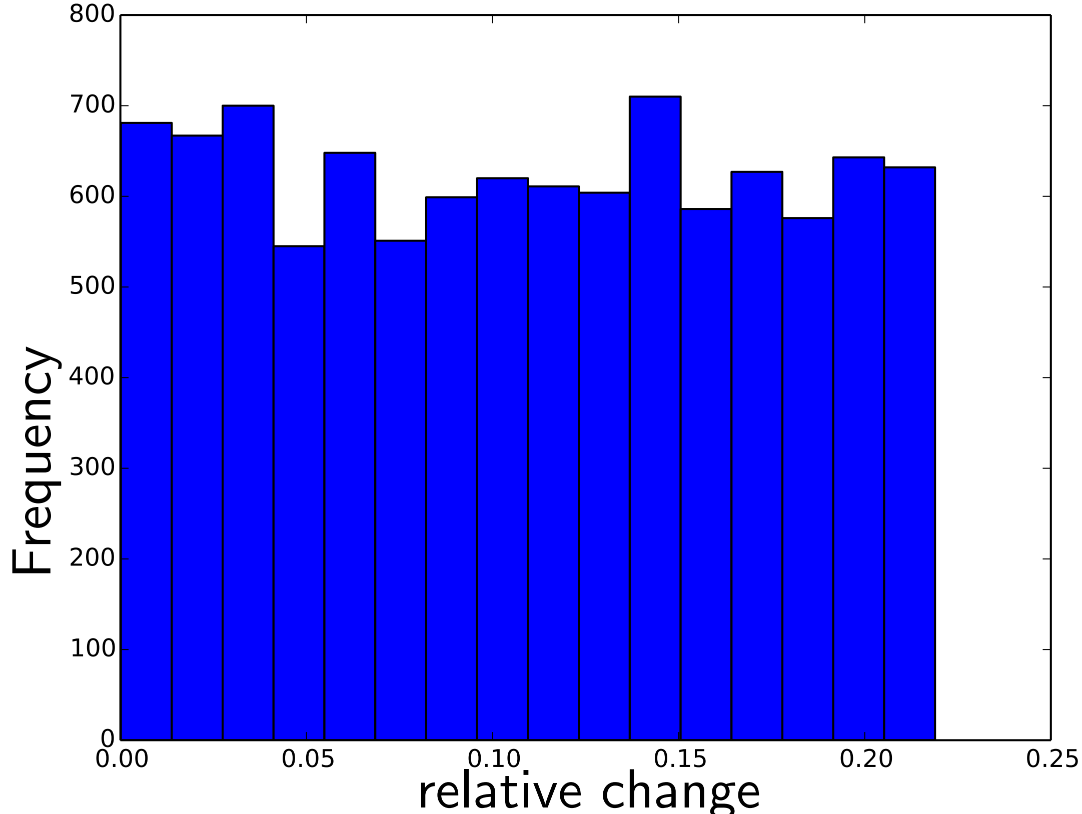
Example failure for linear programming

$$c = \begin{bmatrix} 100 \\ 199.9 \\ -5500 \\ -6100 \end{bmatrix} \quad A = \begin{bmatrix} -.01 & -.02 & .5 & .6 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 90 & 100 \\ 0 & 0 & 40 & 50 \\ 100 & 199.9 & 700 & 800 \\ & -I_4 & & \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 1000 \\ 2000 \\ 800 \\ 100000 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

c vector of costs/profits for two drugs, constraints $Ax \preceq b$ on production

- what happens if we vary percentages $.01$, $.02$ (chemical composition of raw materials) by .5% and 2%, i.e. $.01 \pm .00005$ and $.02 \pm .0004$?

Example failure for linear programming



Frequently lose 15–20% of profits

Alternative robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & (A + \Delta)x \preceq b, \quad \text{all } \Delta \in \mathcal{U} \end{array}$$

where $|\Delta_{11}| \leq .00005$, $|\Delta_{12}| \leq .0004$, $\Delta_{ij} = 0$ otherwise

- solution x_{robust} has degradation *provably* no worse than 6%

How to choose uncertainty sets

- uncertainty set $\mathcal{U}$ a modeling choice
- common idea: let U be random variable, want constraints that

$$\mathbf{prob}(f_i(x,U) \geq 0) \leq \epsilon \tag{1}$$

- typically hard (non-convex except in special cases)
- find set $\mathcal{U}$ such that $\mathbf{prob}(U \in \mathcal{U}) \geq 1 - \epsilon$, then sufficient condition for (1)

$$f_i(x,u) \leq 0 \quad \text{for all } u \in \mathcal{U}$$

Uncertainty set with Gaussian data

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x > b_i) \leq \epsilon, \;\; i = 1, \ldots, m \end{array}$$

coefficient vectors a_i i.i.d. $\mathcal{N}(\overline{a}, \Sigma)$ and failure probability ϵ

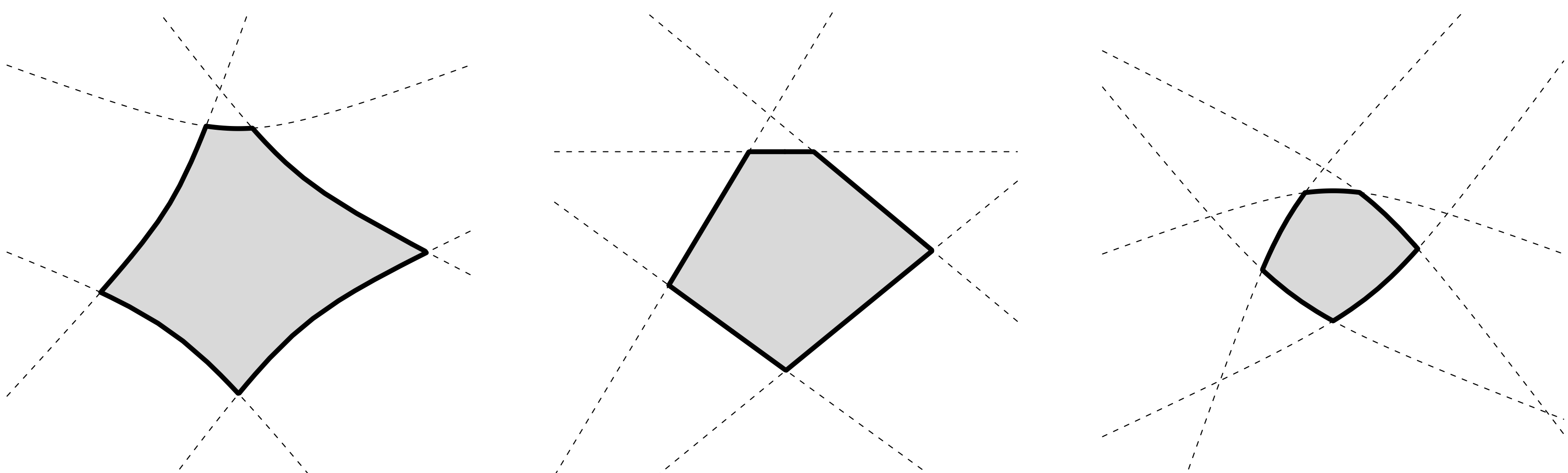
- marginally $a_i^T x \sim \mathcal{N}(\overline{a}_i^T x, x^T \Sigma x)$
- for $\epsilon = .5$, just LP

$$\text{minimize} \;\; c^T x \;\; \text{subject to} \;\; a_i^T x \leq b_i, \;\; i = 1, \ldots, m$$

- what about $\epsilon = .1, .9$?

Gaussian uncertainty sets

$$\{x \mid \mathbf{prob}(a_i^T x > b_i) \leq \epsilon\} = \{x \mid \overline{a}_i^T x - b_i - \Phi^{-1}(\epsilon)\sqrt{x^T \Sigma x} \leq 0\}$$



$\epsilon = .9$ $\epsilon = .5$ $\epsilon = .1$

Problem is convex, so no problem?

not quite...
consider quadratic constraint

$$\|Ax + Bu\|_2 \leq 1 \;\text{ for all }\; \|u\|_\infty \leq 1$$

- convex quadratic *maximization* in u
- solutions on extreme points $u \in \{-1, 1\}^n$
- and NP-hard to maximize (even approximately [Håstad]) convex quadratics over hypercube

Robust LPs

Important question: when is a robust LP still an LP (robust SOCP an SOCP, robust SDP an SDP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & (A+U)x \preceq b \;\; \text{for } U \in \mathcal{U}. \end{array}$$

can always represent formulation constraint-wise, consider only one inequality

$$(a+u)^T x \leq b \;\; \text{for all } u \in \mathcal{U}.$$

- Simple example: $\mathcal{U} = \{u \in \mathbf{R}^n \mid \|u\|_\infty \leq \delta\}$, then

$$a^T x + \delta \|x\|_1 \leq b$$

Polyhedral uncertainty

for matrix $F \in \mathbf{R}^{m \times n}$, $g \in \mathbf{R}^m$,

$$(a+u)^T x \leq b \quad \text{for } u \in \mathcal{U} = \{u \in \mathbf{R}^n \mid Fu + g \succeq 0\}.$$

duality essential for transforming (semi-)infinite inequality into tractable problem

- Lagrangian for maximizing $u^T x$:

$$L(u, \lambda) = x^T u + \lambda^T (Fu + g), \quad \sup_u L(u, \lambda) = \begin{cases} +\infty & \text{if } F^T \lambda + x \neq 0 \\ \lambda^T g & \text{if } F^T \lambda + x = 0. \end{cases}$$

- gives equivalent inequality constraints

$$a^T x + \lambda^T g \leq b, \quad F^T \lambda + x = 0, \quad \lambda \succeq 0.$$

Portfolio optimization (with robust LPs)

- n assets $i = 1, \ldots, n$, random multiplicative return R_i with $\mathbf{E}[R_i] = \mu_i \geq 1$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$

- "certain" problem has solution $x_{\rm nom} = e_1$,

$$\begin{array}{ll} \text{maximize} & \mu^T x \\ \text{subject to} & x^T \mathbf{1} = 1, \ x \succeq 0 \end{array}$$

- if asset i varies in range $\mu_i \pm u_i$, robust problem

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^n \inf_{u \in [-u_i, u_i]} (\mu_i + u) x_i \\ \text{subject to} & \mathbf{1}^T x = 1, \ x \succeq 0 \end{array}$$

and equivalent

$$\begin{array}{ll} \text{maximize} & \mu^T x - u^T x \\ \text{subject to} & \mathbf{1}^T x = 1, \ x \succeq 0 \end{array}$$

Robust LPs as SOCPs

norm-based uncertainty on data vectors a,

$$(a + Pu)^T x \le b \quad \text{for } u \in \mathcal{U} = \{u \in \mathbf{R}^m \mid \|u\| \le 1\},$$

gives dual-norm constraint

$$a^T x + \left\|P^T x\right\|_* \le b$$

Portfolio optimization (tigher control)

- Returns $R_i \in [\mu_i - \mathsf{u}_i, \mu_i + \mathsf{u}_i]$ with $\mathbf{E}\, R_i = \mu_i$
- guarantee return with probability $1 - \epsilon$

$$\underset{\mu,t}{\text{maximize}} \;\; t \;\; \text{subject to} \;\; \mathbf{prob}\left(\sum_{i=1}^{n} R_i x_i \geq t\right) \geq 1 - \epsilon$$

- *value at risk* is non-convex in x, approximate it?
- approximate with high-probability bounds
- less conservative than LP (certain returns) approach

Portfolio optimization: probability approximation

- Hoeffding's inequality

$$\mathbf{prob}\left(\sum_{i=1}^n (R_i - \mu_i)x_i \leq -t\right) \leq \exp\left(-\frac{t^2}{2\sum_{i=1}^n x_i^2 \mathsf{u}_i^2}\right).$$

- written differently

$$\mathbf{prob}\left[\sum_{i=1}^n R_i x_i \leq \mu^T x - t\left(\sum_{i=1}^n \mathsf{u}_i^2 x_i^2\right)^{\frac{1}{2}}\right] \leq \exp\left(-\frac{t^2}{2}\right)$$

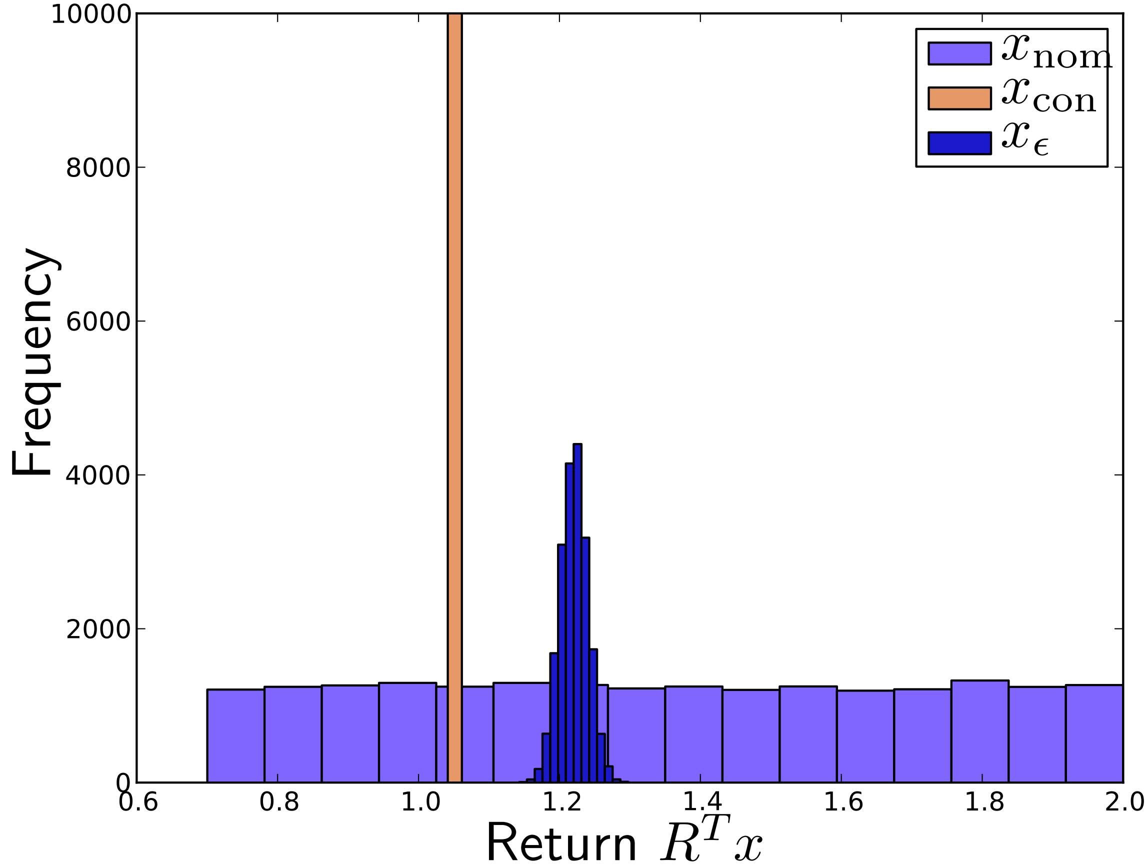
- set $t = \sqrt{2\log(1/\epsilon)}$, gives robust problem

$$\text{maximize} \;\; \mu^T x - \sqrt{2\log\frac{1}{\epsilon}}\,\|\mathbf{diag}(u)x\|_2 \;\; \text{subject to } \mathbf{1}^T x = 1,\; x \succeq 0.$$

Portfolio optimization comparison

- data $\mu_i = 1.05 + \frac{3(n-i)}{10n}$, uncertainty $|u_i| \leq \mathsf{u}_i = .05 + \frac{n-i}{2n}$ and $\mathsf{u}_n = 0$
- nominal minimizer $x_{\rm nom} = e_1$
- conservative (LP) minimizer $x_{\rm con} = e_n$ (guaranteed 5% return),
- robust (SOCP) minimizer x_ϵ for value-at risk $\epsilon = 2 \times 10^{-4}$

Portfolio optimization comparison



Returns chosen randomly in $\mu_i \pm \mathrm{u}_i$, 10,000 experiments

LPs with conic uncertainty

- convex cone K, dual cone $K^* = \{v \in \mathbf{R}^m \mid v^T x \geq 0, \text{ all } x \in K\}$
- recall $x \succeq_K y$ iff $x - y \in K$
- robust inequality

$$(a+u)^T x \leq b \quad \text{for all } u \in \mathcal{U} = \{u \in \mathbf{R}^n \mid Fu + g \succeq_K 0\}$$

- under constraint qualification, equivalent to

$$a^T x + \lambda^T g \leq b, \quad \lambda \succeq_{K^*} 0, \quad x + F^T \lambda = 0$$

Example calculation: LP with semidefinite uncertainty

- symmetric matrices $A_0, A_1, \ldots, A_m \in \mathbf{S}^k$, robust counterpart to $a^T x \leq b$

$$(a + Pu)^T x \leq b \quad \text{for all } u \text{ s.t. } A_0 + \sum_{i=1}^m u_i A_i \succeq 0$$

- cones $K = \mathbf{S}_+^k$, $K^* = \mathbf{S}_+^k$
- Slater condition: $\bar{u}$ such that $A_0 + \sum_i A_i \bar{u}_i \succ 0$
- duality gives equivalent representation

$$a^T x + \mathbf{tr}(\Lambda A_0) \leq b, \quad P^T x + \begin{bmatrix} \mathbf{tr}(\Lambda A_1) \\ \vdots \\ \mathbf{tr}(\Lambda A_m) \end{bmatrix} = 0, \quad \Lambda \succeq 0.$$

Robustness sets for distributions

- stochastic optimization problem with scenarios $\omega_1, \ldots, \omega_k$
- convex objectives $f(x, \omega_i)$
- probabilities $\pi_i = \mathbf{prob}(\omega = \omega_i)$ with $\sum_{i=1}^k \pi_i = 1$
- basic stochastic programming problem

$$\text{minimize} \quad F_0(x) = \mathbf{E}\, f(x, \omega) = \sum_{i=1}^k \pi_i f(x, \omega_i)$$

given N Monte-Carlo (or other) observations of ω, Monte-Carlo problem:

$$\text{minimize} \quad \hat{F}_0(x) = \sum_{i=1}^k \hat{\pi}_i f(x, \omega_i)$$

- goal: get a (high probability) certificate of solution

Confidence sets for a distribution

- can use probabilistic convergence results to get uncertainty sets
- divergences: ϕ-divergence between probability vectors $p, q \in \mathbf{R}^k_+$ with $\mathbf{1}^T p = \mathbf{1}^T q = 1$:

$$D_\phi(p,q) = \sum_{i=1}^k \phi\left(\frac{p_i}{q_i}\right) q_i$$

  for $\phi : \mathbf{R}_+ \to \mathbf{R}$ convex with $\phi(1) = 0$

- statistics fact: if $\phi''(1) = 1$, then for

$$\hat{p}^N = \left(\frac{\#\{\omega = \omega_1\}}{N}, \ldots, \frac{\#\{\omega = \omega_k\}}{N}\right),$$

  have convergence

$$\mathbf{prob}\left(D_\phi(p, \hat{p}_N) \leq \frac{u}{N}\right) \to \mathbf{prob}\left(\|Z\|_2^2 \leq u\right) \quad \text{for} \quad Z \sim \mathcal{N}(0, I_{k-1})$$

Confidence sets for a distribution (continued)

- to get (asymptotic) probability ϵ of failure, choose $u = u_{k,\epsilon}$ such that

$$\mathbf{prob}\left(\|Z\|_2^2 \leq u\right) = 1 - \epsilon$$

- uncertainty set on probabilities

$$\mathcal{P}_{N,\epsilon} = \left\{\pi \in \mathbf{R}_+^k \mid \mathbf{1}^T\pi = 1,\ D_\phi(\pi, \hat{\pi}^N) \leq \frac{u_{k,\epsilon}}{N}\right\}$$

- robust optimization problem:

$$\text{minimize} \quad \sup_{\pi \in \mathcal{P}_N} \sum_{i=1}^k \pi_i f(x, \omega_i)$$

Duality for uncertainty divergence-based uncertainty sets

- duality gives that

$$\sup_{\pi} \left\{ \pi^T t \mid \pi^T \mathbf{1} = 1, \pi \succeq 0, \sum_{i=1}^{k} p_i \phi\left(\frac{\pi_i}{p_i}\right) \leq u \right\}$$

$$= \inf_{\lambda \geq 0, \eta} \left\{ \lambda \sum_{i=1}^{k} \phi^* \left(\frac{t_i - \eta}{\lambda}\right) + \lambda u + \eta \right\}$$

- equivalent robust optimization problem:

$$\text{minimize} \quad \lambda \sum_{i=1}^{k} \hat{\pi}_i \phi^* \left(\frac{f(x, \omega_i) - \eta}{\lambda}\right) + \frac{\lambda u_{k,\epsilon}}{N} + \eta$$

in variables x, λ, η with implicit constraint $\lambda > 0$

Example: quadratic uncertainty sets

- for function $\phi(t) = \frac{1}{2}(t-1)^2$, get problem with SOCP constraints:

$$
\begin{array}{ll}
\text{minimize} & \hat{\pi}^T v + \left(\frac{1}{N} u_{k,\epsilon} + \frac{1}{2}\right)\lambda + t \\
\text{subject to} & f(x,\omega_i) - t \leq y_i,\ y_i \geq 0 \ \text{ for } i = 1,\ldots,k \\
& \frac{y_i^2}{\lambda} \leq 2v_i
\end{array}
$$

in variables $v \in \mathbf{R}^k_+, y \in \mathbf{R}^k, x \in \mathbf{R}^n, \lambda \geq 0, t \in \mathbf{R}$

Example: quadratic uncertainty for least squares

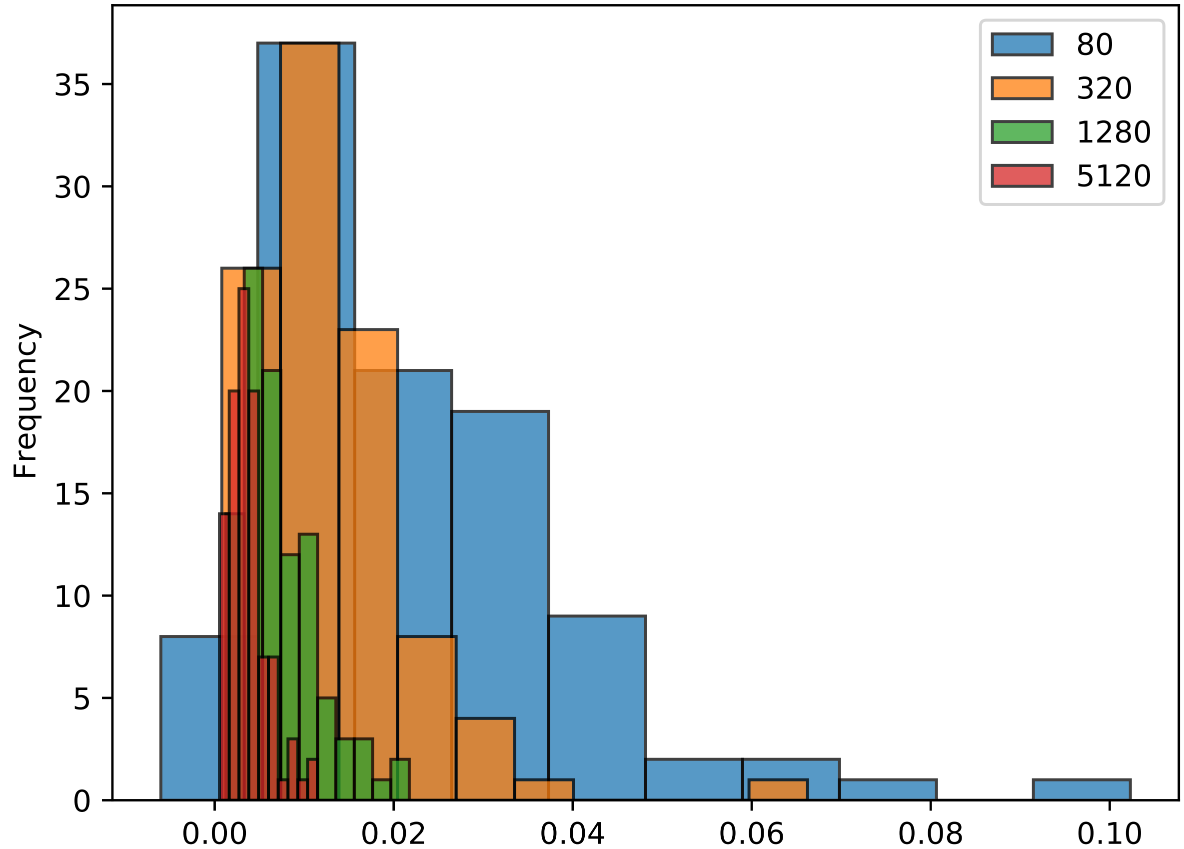
- have $k = 10$ scenarios as vectors $(a_i, b_i) \in \mathbf{R}^n \times \mathbf{R}$ with $n = 3$
- objectives $f(x, (a, b)) = (a^T x - b)^2$
- use χ^2-approximation for different sample sizes $N = 80, 320, 1280, 5120$
- choose $\pi_{\text{true}} \in \mathbf{R}^k_+$, $\pi_{\text{true}}^T \mathbf{1} = 1$ randomly in 100 different experiments
- for uncertainty set

$$\mathcal{P}_{N,\epsilon} = \left\{ \pi \in \mathbf{R}^k_+ \mid \pi^T \mathbf{1} = 1, \sum_{i=1}^k \hat{\pi}_i \left(\frac{\pi_i}{\hat{\pi}_i} - 1 \right)^2 \leq \frac{u_{k,\epsilon}}{N} \right\}$$

choose $\hat{x}^{N,\epsilon}$ to solve

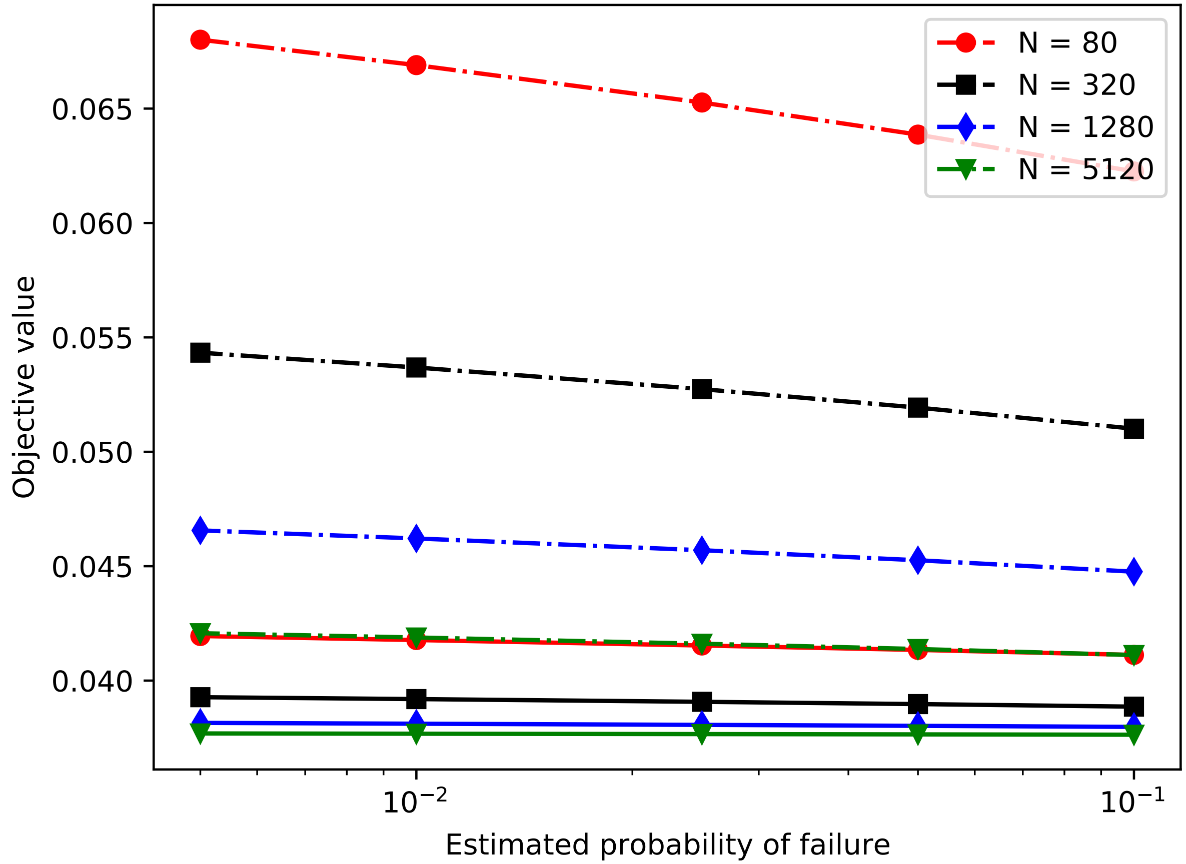
$$\text{minimize} \quad \hat{F}_0(x) = \sup_{\pi \in \mathcal{P}_{N,\epsilon}} \sum_{i=1}^k \pi_i (a_i^T x - b_i)^2$$

Histogram of estimated versus true objective values



- histogram of gaps $\hat{F}_0(\hat{x}) - F_0(\hat{x})$ at confidence level $\epsilon = .05$
- colors for sample sizes $N = 80, 320, 1280, 5120$

Plots of estimated and true objective values



- estimated objective values $\hat{F}_0(\hat{x})$ versus level ϵ

Robust second-order cone problems

- Lorentz/SOCP cone, nominal inequality

$$\|Ax + b\|_2 \leq c^T x + d$$

- $A = [a_1 \ \cdots \ a_n]^T \in \mathbf{R}^{m \times n}$, allow A, c to vary

- interval uncertainty

- ellipsoidal uncertainty

- matrix uncertainty

SOCPs with interval uncertainty

entries A_{ij} perturbed by Δ_{ij} with $|\Delta_{ij}| \leq \delta$, c by cone:

$$\|(A+\Delta)x+b\|_2 \leq (c+u)^T x + d \;\; \text{all} \;\; \|\Delta\|_\infty \leq \delta, \; u \in \mathcal{U}$$

- split into two inequalities (first is robust LP)

$$\|(A+\Delta)x+b\|_2 \leq t, \quad t \leq (c+u)^T x + d$$

second

$$\begin{aligned}
\sup_{\Delta:|\Delta_{ij}|\leq\delta} \|(A+\Delta)x+b\|_2 &= \sup_{\Delta:|\Delta_{ij}|\leq\delta} \left(\sum_{i=1}^m [(a_i+\Delta_i)^T x + b_i]^2 \right)^{1/2} \\
&= \sup_{\Delta \in \mathbf{R}^{m\times n}} \left\{ \|z\|_2 \mid z_i = a_i^T x + \Delta_i^T x + b_i, \|\Delta_i\|_\infty \leq \delta \right\} \\
&= \inf \left\{ \|z\|_2 \mid z_i \geq |a_i^T x + b| + \delta \|x\|_1 \right\}.
\end{aligned}$$

SOCPs with ellipse-like uncertainty

- matrices $P_1, \ldots, P_m \in \mathbf{R}^{n \times n}$, $u \in \mathbf{R}^m$ with $\|u\| \leq 1$
- robust/uncertain inequality

$$\left(\sum_{i=1}^{m} [(a_i + P_i u)^T x + b_i]^2 \right)^{1/2} \leq t \quad \text{for all } u \text{ s.t. } \|u\|_2 \leq 1.$$

- rewrite $z_i \geq \sup_{\|u\| \leq 1} |a_i^T x + b_i + u^T P_i^T x|$, equivalent

$$\|z\|_2 \leq t, \quad z_i \geq |a_i^T x + b_i| + \left\| P_i^T x \right\|_*, \quad i = 1, \ldots, m.$$

SOCPs with matrix uncertainty

- Matrix $P \in \mathbf{R}^{m \times n}$ and radius δ, uncertain inequality

$$\|(A + P\Delta)x + b\|_2 \leq t, \quad \text{for } \Delta \in \mathbf{R}^{n \times n} \text{ s.t. } \|\Delta\| \leq \delta,$$

- tool one: Schur complements gives equivalence of

$$\|x\|_2 \leq t \quad \text{and} \quad \begin{bmatrix} t & x^T \\ x & tI_n \end{bmatrix} \succeq 0.$$

- tool two: homogeneous S-lemma

$$x^T A x \geq 0 \text{ implies } x^T B x \geq 0 \quad \text{if and only if} \quad \exists\, \lambda \geq 0 \text{ s.t. } B \succeq \lambda A.$$

SOCPs with matrix uncertainty

$$\|(A + P\Delta)x + b\|_2 \leq t, \quad \text{for } \Delta \in \mathbf{R}^{n\times n} \text{ s.t. } \|\Delta\| \leq \delta,$$

equivalent to

$$\begin{bmatrix} t & ((A+P\Delta)x+b)^T \\ (A+P\Delta)x+b & tI_m \end{bmatrix} \succeq 0 \quad \text{for} \quad \|\Delta\| \leq 1.$$

or

$$ts^2 + 2s((A+P\Delta)x+b)^T v + t\,\|v\|_2^2 \geq 0 \quad \text{for all } s \in \mathbf{R}, v \in \mathbf{R}^m, \quad \|\Delta\| \leq 1.$$

SOCPs with matrix uncertainty: final result

$$\|(A + P\Delta)x + b\|_2 \leq t, \quad \text{for } \Delta \in \mathbf{R}^{n \times n} \text{ s.t. } \|\Delta\| \leq \delta,$$

equivalent to

$$\begin{bmatrix} t & (Ax+b)^T & x^T \\ Ax+b & t - \lambda P P^T & 0 \\ x & 0 & \lambda I_n \end{bmatrix} \succeq 0.$$

Example: robust regression

$$\text{minimize } \|Ax - b\|_2$$

where A corrupted by Gaussian noise,

$$A = A_\star + \Delta \quad \text{for} \quad \Delta_{ij} \sim \mathcal{N}(0,1)$$

decide to be robust to Δ by

- bounding individual entries Δ_{ij}
- bounding norms of rows Δ_i
- bounding (ℓ_2-operator) norm of Δ

Choice of uncertainty in robust regression

Theorem [e.g. Vershynin 2012] Let $\Delta \in \mathbf{R}^{m\times n}$ have i.i.d. $\mathcal{N}(0,1)$ entries. For all $t \geq 0$, the following hold:

- For each pair i, j

$$\mathbf{prob}(|\Delta_{ij}| \geq t) \leq 2\exp\left(-\frac{t^2}{2}\right).$$

- For each i

$$\mathbf{prob}(\|\Delta_i\|_2 \geq \sqrt{n} + t) \leq \exp\left(-\frac{t^2}{2}\right).$$

- For the entire matrix Δ,

$$\mathbf{prob}(\|\Delta\| \geq \sqrt{m} + \sqrt{n} + t) \leq \exp\left(-\frac{t^2}{2}\right).$$

Choice of uncertainty in robust regression

idea: choose bounds $t(\delta)$ to guarantee $\mathbf{prob}(\text{deviation} \geq t(\delta)) \leq \delta$

- coordinate-wise: $t_\infty(\delta)^2 = 2\log\frac{2mn}{\delta}$,

$$\mathbf{prob}(\max_{i,j} |\Delta_{ij}| \geq t_\infty(\delta)) \leq 2mn \exp\left(-\frac{t_\infty(\delta)^2}{2}\right) = \delta$$

- row-wise: $t_2(\delta)^2 = 2\log\frac{m}{\delta}$,

$$\mathbf{prob}(\max_{i} \|\Delta_i\|_2 \geq t_2(\delta)) \leq m \exp\left(-\frac{t_2(\delta)^2}{2}\right) = \delta$$

- matrix-norm: $t_{\rm op}(\delta)^2 = 2\log\frac{1}{\delta}$,

$$\mathbf{prob}(\|\Delta\| \geq \sqrt{n} + \sqrt{m} + t_{\rm op}(\delta)) \leq \exp\left(-\frac{t_{\rm op}(\delta)^2}{2}\right) = \delta.$$

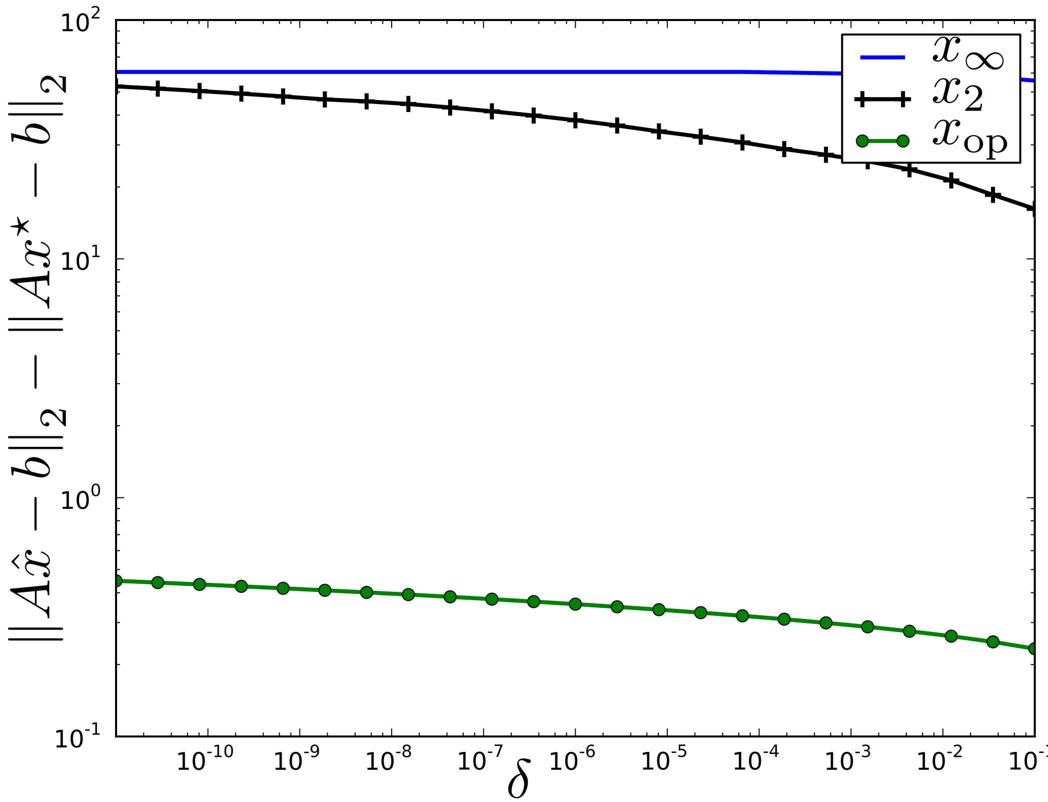
Robust regression results

$$\underset{x}{\text{minimize}} \quad \sup_{\Delta \in \mathcal{U}} \|(A + \Delta)x - b\|_2$$

where $\mathcal{U}$ is one of the three uncertainty sets

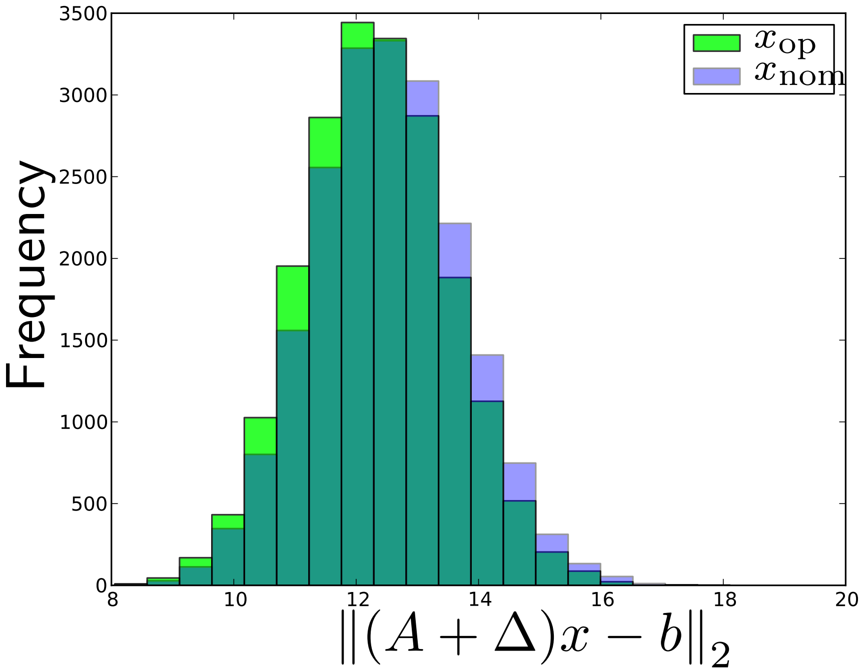
$$\begin{aligned}
\mathcal{U}_\infty &= \{\Delta \mid \|\Delta\|_\infty \leq t_\infty(\delta)\}, \\
\mathcal{U}_2 &= \{\Delta \mid \|\Delta_i\|_2 \leq \sqrt{n} + t_2(\delta) \text{ for } i = 1, \ldots m\}, \\
\mathcal{U}_{\text{op}} &= \{\Delta \mid \|\Delta\| \leq \sqrt{n} + \sqrt{m} + t_{\text{op}}(\delta)\}.
\end{aligned}$$

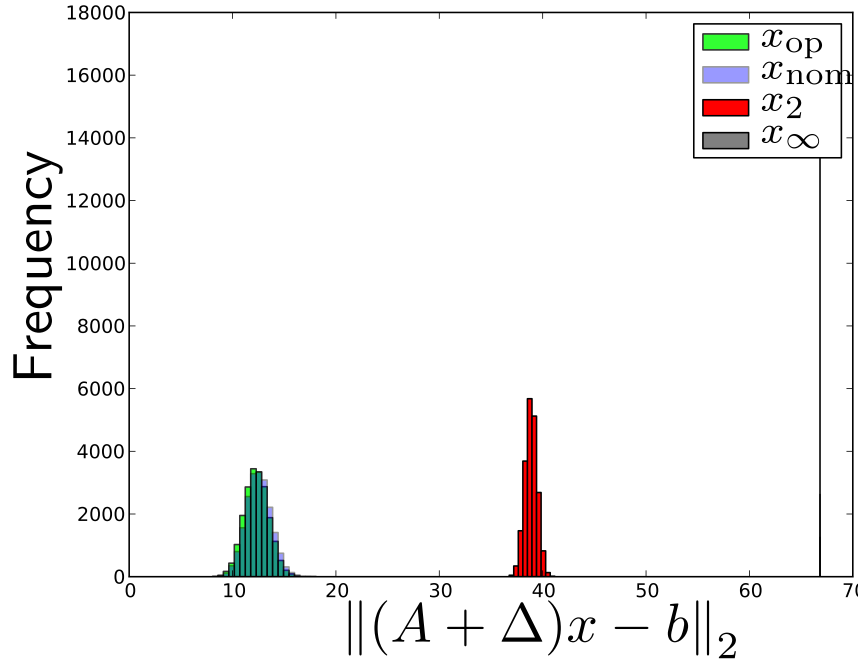
Robust regression results



Objective value $\|A\hat{x} - b\|_2 - \|Ax^\star - b\|_2$ versus δ, where $x^\star$ minimizes nominal objective and $\hat{x}$ denotes robust solution

Robust regression results





- residuals for the robust least squares problem $\|(A+\Delta)x-b\|_2$
- uncertainty sets $\mathcal{U}_{\rm nom} = \{0\}$ vs. $\mathcal{U}_\infty$, $\mathcal{U}_2$, $\mathcal{U}_{\rm op}$
- experiment with $N = 10^5$ random Gaussian matrices