Convex Optimization EE364a: Review Session 1
Notation and Convex Sets

Stanford University

Winter Quarter, 1/15/2013
Outline

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General sets

Norms

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Convexity
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Administration Overview

- Office hours, an unstructured time to ask questions:
  - Monday 4:15-6:15
  - Tuesday 4:15-6:15
  - Wednesday 2:15-4:15, 4:15-6:15
  - Thursday 12:50-2:50, 4:15-6:15, 6:15-8:15

- Review session, more structured time in which we will review specific topics:
  - Tuesday 2:15-3:05

- We also have a site set up on Piazza, where you can post questions

- Contact TAs through:
  ee364a-win1213-staff@lists.stanford.edu

- Homeworks are due across from Packard 243 on Fridays at 5pm
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Notation — Basic symbols

\( x \in \):

- \( \mathbb{Z} \): \( x \) is an integer
- \( \mathbb{R} \): \( x \) is a real scalar
- \( \mathbb{R}^+ \): \( x \) is a real scalar \( \geq 0 \)
- \( \mathbb{R}^{++} \): \( x \) is a real scalar \( > 0 \)
- \( \mathbb{R}^n \): \( x \) is real vector of length \( n \)
- \( \mathbb{C}^n \): \( x \) is complex vector of length \( n \)

\( X \in \):  

- \( \mathbb{R}^{n \times m} \): \( X \) is a matrix of reals of size \( n \times m \)
- \( \mathbb{S}^n \): \( X \) is a symmetric matrix of size \( n \times n \)
- \( \mathbb{S}^{n+} \): \( X \) is a positive semidefinite matrix of size \( n \times n \)
- \( \mathbb{S}^{n++} \): \( X \) is a positive definite matrix of size \( n \times n \)
Notation — Functions

\[ f : \mathbb{R}^n \rightarrow \mathbb{R}^m, \ \text{dom} \ f = C \]

- We say that \( f \) is an \( \mathbb{R}^m \)-valued function on domain \( C \) a subset of \( \mathbb{R}^n \)
- Function \( f \) takes as input a vector of reals, \( x \), of length \( n \) such that \( x \in C \) and returns a vector of reals of length \( m \)

\( \sqrt{\text{sqrt}} \) (the square root function) would be expressed:

- \( \sqrt{\text{sqrt}} : \mathbb{R} \rightarrow \mathbb{R}, \ \text{dom} \ \sqrt{\text{sqrt}} = \mathbb{R}_+ \)
- \( \sqrt{\text{sqrt}} : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \)
Notation – Sets

\[ C = \{ x \in \mathbb{R}^n \mid f_i(x) \leq b_i, i = 0, \ldots, m \} \]

► We might read this:
   The set \( C \) consists of all \( x \) in \( \mathbb{R}^n \) such that \( f_i(x) \leq b_i \) for \( i \) from 0 to \( m \)

► This means that an element \( x \in \mathbb{R}^n \) is in the set \( C \) if all of the \( m \) inequalities to the right of \( \mid \) evaluate to true

\[ C = \{ g(x) \mid f_i(x) \leq b_i, i = 0, \ldots, m \}, f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \]

► We might read this:
   The set \( C \) consists of all values \( g(x) \) such that \( f_i(x) \leq b_i \) for \( i \) from 0 to \( m \)

► This means that an element \( y \in \mathbb{R}^3 \) is in the set \( C \) if there exists an \( x \in \mathbb{R}^2 \) such that \( y = g(x) \) and, for that \( x \), all of the \( m \) inequalities to the right of \( \mid \) evaluate to true

► This is NOT a set of functions, but \( C \subseteq \mathbb{R}^3 \)
Notation — Representing sets

Most set representations are not unique:

- \( C = \{ x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 4 \} \)
- \( C = \{ (r \cos(\theta), r \sin(\theta)) \mid 0 \leq r \leq 2, \theta \in [0, 2\pi] \} \)
- \( C = \{ x \in \mathbb{R}^2 \mid x^T \left( \frac{1}{4} I \right) x \leq 1 \} \)
- \( C = \{ 2u \mid \|u\|_2 \leq 1, u \in \mathbb{R}^2 \} \)
- \( C = \left\{ 2\sqrt{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} u \mid \|u\|_2 \leq 1, u \in \mathbb{R}^2 \right\} \)
- \( C = \bigcap_{\theta \in [0, 2\pi]} \{ x \mid (\cos(\theta), -\sin(\theta))^T x \leq 2 \} \)
Notation — Set operators

- \( A = B \) (equality)
  \( x \in A \) if and only if \( x \in B \)

- \( A \subseteq B \) (subset)
  If \( x \in A \) then \( x \in B \)

- \( C = A \cup B \) (union)
  \( C = \{ x \mid x \in A \text{ or } x \in B \} \)

- \( C = A \cap B \) (intersection)
  \( C = \{ x \mid x \in A, x \in B \} \)

- \( C = A - x \) (operation by an element)
  \( C = \{ y - x \mid y \in A \} \)

- \( C = A \setminus B \) (set difference)
  \( C = \{ x \in A \mid x \notin B \} \)

- \( C = A + B \) (Minkowski sum)
  \( C = \{ x + y \mid x \in A, y \in B \} \)
Set operators — Simple examples

\[ A = \{1, 2, 3\}, \ B = \{3, 4, 5\} \]

- \[ A \cup B = \{1, 2, 3, 4, 5\} \]
- \[ A \cap B = \{3\} \]
- \[ A - 1 = \{0, 1, 2\} \]
- \[ A \setminus B = \{1, 2\} \]
- \[ B \setminus A = \{4, 5\} \]
- \[ A + B = \{4, 5, 6, 7, 8\} \]
- \[ (A - 1) \cap B = \emptyset \]
Set operators — Minkowski sum and set difference

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Set interior

- An element $x \in C \subseteq \mathbb{R}^n$ is called an interior point of $C$ if there exists an $\epsilon > 0$ for which $\{y \mid \|y - x\|_2 \leq \epsilon\} \subseteq C$
- The set of all points interior to $C$ is called the interior of $C$ and is denoted $\text{int} \, C$
- If $C = \text{int} \, C$, then the set is open
- A set is closed if its complement, $\mathbb{R}^n \setminus C$, is open
Set interior — Simple examples

- Given $C = \{ x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 4 \}$
  $\text{int } C = \{ x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 4 \}$

- Given $C = \{ x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq 4, x_3 = 0 \}$
  $\text{int } C = \emptyset$
Set boundary

- The closure of a set $\mathcal{C} \subseteq \mathbb{R}^n$ is defined as
  $$\text{cl} \mathcal{C} = \mathbb{R}^n \setminus \text{int}(\mathbb{R}^n \setminus \mathcal{C})$$
- The boundary of $\mathcal{C}$ is defined as $\text{bd} \mathcal{C} = \text{cl} \mathcal{C} \setminus \text{int} \mathcal{C}$
Set boundary — Simple examples

Given $C = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 4\}$,
$\text{bd } C = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 4\}$

Given $C = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq 4, x_3 = 0\}$,
$\text{bd } C = C$
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A function $f : \mathbb{R}^n \to \mathbb{R}$ with $\text{dom } f = \mathbb{R}^n$ is called a norm iff:

- $f(x) = 0$ if and only if $x = 0$
- $f(tx) = |t|f(x)$
- $f(x + y) \leq f(x) + f(y)$
- $f \geq 0$

Norms on $\mathbb{R}^n$

- $\|x\|_1 = \sum_{i=1}^{n} |x_i|$, ($\ell_1$-norm)
- $\|x\|_2 = \left( \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$, ($\ell_2$-norm, Euclidean norm)
- $\|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$, $p \geq 1$, ($p$-norm)
- $\|x\|_\infty = \max_i |x_i|$, (infinity norm, Chebyshev norm)
Norm balls

A norm ball is \( \{ x \mid \| x - x_c \| \leq r \} \)

Here are some balls in \( \mathbb{R}^2 \) with \( x_c = 0 \) and \( r = 1 \), which norm induces them?:

\[
\| \cdot \|_2 \quad \| \cdot \|_\infty \quad \| \cdot \|_1 \quad \| \cdot \|_{3.5}
\]
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Supremum and infimum

The supremum of a set \( C \) is the least upper bound of the set \( C \)

- \( a \) is an upper bound of \( C \) if \( x \leq a, \ \forall x \in C \)
- We represent this as \( \sup C \)
- For a finite set \( C \), \( \sup C = \max C \)
- The supremum of a set, need not be in the set:
  \[
  \sup \{x \in \mathbb{R} \mid x < 2\} = 2
  \]
- \( \sup \emptyset = -\infty \), \( \sup C = \infty \) if \( C \) is unbounded above

The infimum of a set \( C \) is the greatest lower bound of the set \( C \)

- \( a \) is a lower bound of \( C \) if \( x \geq a, \ \forall x \in C \)
- We represent this as \( \inf C \)
- For a finite set \( C \), \( \inf C = \min C \)
- \( \inf C = -\sup (-C) \)
- \( \inf \emptyset = \infty \), \( \inf C = -\infty \) if \( C \) is unbounded below
Minimum and minimal

- Remember:
  - A generalized inequality is defined by a proper cone (convex, closed, solid, pointed) $K$
  - $x \preceq_K y$ means $y - x \in K$
  - $x \prec_K y$ means $y - x \in \text{int } K$
  - This need not be, and often will not be, a total ordering

- If $x$ is a minimum element of $S$ then all elements in $S$ are larger than $x$

- If $x$ is a minimal element of $S$ then no element in $S$ is smaller than $x$

- In a total ordering, minimum and minimal are the same
Minimum and minimal — Examples

Using the cone $\mathbb{R}_2^+$:

- $x_1$ is a minimum element of $S_1$
- $x_2$ is a minimal element of $S_2$
- Is $x_1$ a minimal element of $S_1$?
  - Yes
- Is $x_2$ a minimum element of $S_2$?
  - No
Minimum and minimal — More examples

Consider the ordering induced by \( K = \mathbb{R}^2_+ \)
\( S = \{(1, 2), (2, 3), (3, 2)\} \)

▷ What are the minimal elements of \( S \)?
  ▷ (1, 2)
▷ Does \( S \) have a minimum element?
  ▷ Yes

\( S = \{(0, 3), (0, 4), (1, 2), (1, 3), (2, 0)\} \)

▷ What are the minimal elements of \( S \)?
  ▷ (0, 3), (1, 2), (2, 0)
▷ Does \( S \) have a minimum element?
  ▷ No
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How to test convexity

- Apply definition
  \[ x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \implies \theta x_1 + (1 - \theta) x_2 \in C \]

- Show the the is set is obtained form simple convex sets by operations that preserve convexity:

<table>
<thead>
<tr>
<th>Simple convex sets</th>
<th>Operations that preserve convexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>halfspaces</td>
<td>intersections</td>
</tr>
<tr>
<td>norm balls</td>
<td>affine functions</td>
</tr>
<tr>
<td>norm cones</td>
<td>perspective function</td>
</tr>
<tr>
<td>polyhedra</td>
<td>linear-fractional functions</td>
</tr>
</tbody>
</table>

- Other methods you will learn later
Some simple sets

Are the following sets convex?

- $S = \{ x \in \mathbb{R} \mid x^2 \leq 1 \}$
  - Yes

- $S = \{ x \in \mathbb{R} \mid x^2 \geq 1 \}$
  - No

- $S = \{ (x, x^2) \mid x \in \mathbb{R} \}$
  - No

- $S = \{ (x, y) \mid y \geq x^2, x \in \mathbb{R} \}$
  - Yes
The ellipse

How can we show that an ellipse is convex?

- Show that a mixture of two arbitrary points in the ellipse, also lies in the ellipse
- Show that the intersection of the ellipse with an arbitrary line is convex
- Show that the ellipse is an affine transformation of a $\ell_2$ ball
- Show that the ellipse is a norm ball for a particular norm
- Show that it is an intersection of halfspaces
The hyperbolic set

Is the set \( \{x \in \mathbb{R}^2_+ \mid x_1 x_2 \geq 1\} \) convex?

We will prove convexity using the definition:

- Consider two points \((x_1, x_2), (y_1, y_2)\) in the set.
- If \(x \succeq y\) then \(z = \theta x + (1 - \theta)y \succeq y\), so \(z_1 z_2 \geq y_1 y_2 \geq 1\).
- A similar argument holds if \(y \succeq x\).
- If \(x \not\succeq y\) or \(y \not\succeq x\) then \((y_1 - x_1)(y_2 - x_2) < 0\)
  - \((\theta x_1 + (1 - \theta)y_1)(\theta x_2 + (1 - \theta)y_2)\)
  - \(= \theta^2 x_1 x_2 + (1 - \theta)^2 y_1 y_2 + \theta(1 - \theta)x_1 y_2 + \theta(1 - \theta)x_2 y_1\)
  - \(= \theta x_1 x_2 + (1 - \theta)y_1 y_2 - \theta(1 - \theta)(y_1 - x_1)(y_2 - x_2)\)
  - \(\geq 1\).
- Yes, the set is convex.
Copositive matrices

Is the set of copositive matrices $C = \{ M \in \mathbb{R}^{n \times n} \mid x^T M x \geq 0, \forall x \succeq 0 \}$ convex?

- Choose $M_1, M_2 \in C$
- $x^T M_1 x \geq 0$ and $x^T M_2 x \geq 0$ for $x \succeq 0$
- Let $N = \theta M_1 + (1 - \theta) M_2$, $0 \leq \theta \leq 1$
- $x^T N x$
- $= x^T (\theta M_1 + (1 - \theta) M_2) x$
- $= \theta x^T M_1 x + (1 - \theta) x^T M_2 x$
- $\geq 0$, for $x \succeq 0$
- So $N \in C$ so $C$ is convex.