Outline

Announcements

Review of course material

Convex sets
Convex functions
Duality

Non-convex optimization problems

Selected problems from past exams
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Convex sets
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Duality
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Selected problems from past exams
Announcements

- pick up final exams in Bytes Cafe 5pm-5:30pm on 3/15 or 3/16
- normal office hours this week
- posting on Piazza will be disabled on Thursday
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Mathematical optimization problems

(mathematical) optimization problem

minimize $f_0(x)$
subject to $f_i(x) \leq b_i, \quad i = 1, \ldots, m$

- $x = (x_1, \ldots, x_n)$: optimization variables
- $f_0 : \mathbb{R}^n \to \mathbb{R}$: objective function
- $f_i : \mathbb{R}^n \to \mathbb{R}, \quad i = 1, \ldots, m$: constraint functions

optimal solution $x^*$ has smallest value of $f_0$ among all vectors that satisfy the constraints

Review of course material
**Convex optimization problem**

convex optimization problems

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq b_i, \quad i = 1, \ldots, m.
\end{align*}
\]

- objective and constraint functions are convex
- can be solved efficiently and reliably

can be expressed in the form

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad x \in S,
\end{align*}
\]

where \( S \) is a convex set
Convex sets

**convex set**: contains line segment between any two points in the set

\[ x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \implies \theta x_1 + (1 - \theta)x_2 \in C \]

how to show the convexity of a set \( C \)

- apply definition above
- show that \( C \) is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, ...) by operations that preserve convexity
- show that \( C \) is a sublevel set of a convex function
Convex sets

operations that preserve convexity (of sets)

- intersection
- affine functions
- perspective function
- linear-fractional functions

example:

\[
S = \{ x \in \mathbb{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3 \}
\]

where \( p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt \)
Convex functions

$f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\text{dom } f$ is a convex set and

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

for all $x, y \in \text{dom } f$, $0 \leq \theta \leq 1$
Convex functions

how to determine if a function $f$ is convex?

- apply definition
- simple function known to be convex
  - affine, exponential, negative entropy, log-sum-exp, norms, 
    ...
- for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$
- show that $f$ is obtained by operations that preserve convexity
  - nonnegative weighted sum, composition with affine
    function, pointwise maximum and supremum, composition,
    minimization, perspective
Convex functions

pointwise maximum and supremum versus minimization

- if $f_1, \ldots, f_m$ are convex, then $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ is convex
- if $f(x, y)$ is convex in $x$ for each $y \in A$, then

$$g(x) = \sup_{y \in A} f(x, y)$$

is convex
- if $f(x, y)$ is convex in $(x, y)$ and $C$ is a convex set, then

$$g(x) = \inf_{y \in C} f(x, y)$$

is convex
Convex functions

composition of \( g : \mathbb{R}^n \to \mathbb{R} \) and \( h : \mathbb{R} \to \mathbb{R} \):

\[
f(x) = h(g(x))
\]

\( f \) is convex if \( g \) convex, \( h \) convex, \( \tilde{h} \) nondecreasing

\( g \) concave, \( h \) convex, \( \tilde{h} \) nonincreasing

► proof (for \( n = 1 \), differentiable \( g, h \))

\[
f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)
\]

► note: monotonicity must hold for extended-value extension \( \tilde{h} \)
Convex functions

are the following functions convex, concave, or neither?

- $f(x) = \log(a^T x - b)$
  concave: concave function with affine composition

- $f(x) = \log(1 + 1/x)$
  convex: increasing convex function (log-sum-exp) composed with convex function ($-\log(x)$)

- $f^*(y) = \sup_{x \in \text{dom} f} (y^T x - f(x))$
  convex: supremum of an affine function
Duality

**standard form problem** (not necessarily convex)

minimize \( f_0(x) \)

subject to \( f_i(x) \leq 0, \quad i = 1, \ldots, m \)

\( h_i(x) = 0, \quad i = 1, \ldots, p \)

variable \( x \in \mathbb{R}^n \), domain \( \mathcal{D} \), optimal value \( p^* \)

**Lagrangian:** \( L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R} \), with

\( \text{dom} L = \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p \),

\[
L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)
\]

- weighted sum of objective and constraint functions
- \( \lambda_i \) is Lagrange multiplier associated with \( f_i(x) \leq 0 \)
- \( \nu_i \) is Lagrange multiplier associated with \( h_i(x) = 0 \)

Review of course material
Duality

Lagrange dual function: $g : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$,

$$g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu)$$

$g$ is concave, can be $-\infty$ for some $\lambda, \nu$

Lagrange dual problem

$$\begin{align*}
\text{maximize} & \quad g(\lambda, \nu) \\
\text{subject to} & \quad \lambda \succeq 0
\end{align*}$$

- finds best lower bound on $p^*$
- a convex optimization problem; optimal value denoted $d^*$
- $\lambda, \nu$ are dual feasible if $\lambda \succeq 0$, $(\lambda, \nu) \in \text{dom}g$
- often simplified by making implicit constraint $(\lambda, \nu) \in \text{dom}g$ explicit
Duality

facts about duality

- weak duality \((d^* \leq p^*)\) always holds
- strong duality \((d^* = p^*)\) holds in convex problems when constraint qualifications are met
- Slater’s constraint qualification: problem is strictly feasible

\[ \exists x \in \text{int} D : f_i(x) < 0, \quad i = 1, \ldots, m, \quad Ax = b \]

- when strong duality holds, primal and dual optimal points satisfy KKT conditions (sufficient for convex problems)
- dual variables are useful in sensitivity analysis
- theorems of alternatives are used as proof or certificates that a system of equations is infeasible

Review of course material 17
Geometric programming

monomial function

\[ f(x) = cx_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \quad \text{dom}f = \mathbb{R}^n_{++} \]

with \( c > 0 \); exponent \( a_i \) can be any real number

posynomial function: sum of monomials

\[ f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}, \quad \text{dom}f = \mathbb{R}^n_{++} \]

geometric program (GP)

minimize \( f_0(x) \)

subject to \( f_i(x) \leq 1, \quad i = 1, \ldots, m \)

\( h_i(x) = 1, \quad i = 1, \ldots, p \)

with \( f_i \) posynomial, \( h_i \) monomial

Review of course material
Quasiconvex optimization

minimize $f_0(x)$
subject to $f_i(x) \leq 0, \quad i = 1, \ldots, m$
$Ax = b$

with $f_0 : \mathbb{R}^n \to \mathbb{R}$ quasiconvex, $f_1, \ldots, f_m$ convex

- sublevel sets of $f_0$, $S_t = \{x | f_0(x) \leq t\}$, are convex
- transform into feasibility problem
- solve by bisection on $t$
Problems involving cardinality

\[
\begin{align*}
\text{minimize} & \quad \text{card}(x) \\
\text{subject to} & \quad x \in \mathcal{C}
\end{align*}
\]

- cardinality problems are hard to solve exactly
- using simple heuristic, the $l_1$-norm, seems to work well

\[
\begin{align*}
\text{minimize} & \quad \|x\|_1 \\
\text{subject to} & \quad x \in \mathcal{C}
\end{align*}
\]
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2012 question 1

formulation

maximize \( P_1 f_1(x) + \cdots + P_K f_K(x) \)
subject to \( l \leq x \leq u \)

where \( f_1(x) = g(w_k^T x + v_k) \) and \( g(z) = 1/(1 + \exp(-z)) \)

- show the objective need not be quasiconcave
  affine composition preserves quasiconcavity, addition does not
- show that adding the constraint \( w_k^T x + v_k \geq 0 \) makes the problem convex
  the function \( g \) is concave for nonnegative arguments, a nonnegative weighted sum of concave functions is concave, maximizing a concave functions, and constraints are clearly convex
2012 question 2

formulation

\[
\begin{align*}
\text{maximize} \quad & \inf_{\bar{p} \preceq \underline{p} \preceq u} \bar{p}^T x \\
\text{subject to} \quad & 1^T x = 1 \\
\quad & x^T \Sigma x \leq \sigma_{\text{max}}^2
\end{align*}
\]

where \( \Sigma \in S^{n}_{++} \) and \( \sigma_{\text{max}}^2 \in R^{++} \) are given

- is this a convex problem?
  yes, objective is concave (pointwise infimum of affine function of \( x \)) and constraints are clearly convex

- how do we solve it?
  we can give an explicit solution for the objective:
  \[
  \sum_{i=1}^{n} \min(l_i x_i, u_i x_i)
  \]
2012 question 3

formulation

\[ \text{minimize } \left( \frac{1}{N} \right) \sum_{i=1}^{N} \left( d_i^2 - 2d_i (z_i^T P z_i)^{1/2} + z_i^T P z_i \right) \]

with variable \( P \in S_+^n \) and \( z_i = x_i - y_i \)

- is this problem convex?
  yes, first term is constant, the second term is convex (negation of concave function with linear composition), and the third term is linear
2012 question 4

formulation

maximize \[ \sum_{i=1}^{n} \pi_i U \left( \frac{P x_i}{x_i + a_i} \right) \]

subject to \[ P = (1 - c)(1^T a + 1^T x) \]
\[ 1^T x = B, x \succeq 0 \]

with variables \( x \) and \( P \), where \( U(z) \) is a concave increasing function

\[ \text{is this problem convex?} \]

yes, \( P \) is actually a constant because \( 1^T x \) is known and \( U(z) \) is composed with a concave function
2012 question 5

- formulate

\[ AR^k_+ \subseteq BR^p_+ \]

solve as \( k \) feasibility problems

find \( y_i \)

subject to \( a_i = By_i, y_i \geq 0 \)

- formulate

\[ AR^k_+ = \mathbb{R}^n \]

solve as \( n \) feasibility problems

find \( (y_i, z_i) \)

subject to \( Ay_i = e_i, Az_i = -e_i \)

\( y_i \geq 0, z_i \geq 0 \)
2012 question 8

formulation

\[
\text{maximize} \quad \lambda_{\min}(H + \alpha \nu^{\text{nom}}(\nu^{\text{nom}})^T) - \lambda \\
\text{subject to} \quad H\nu^{\text{nom}} = \lambda \nu^{\text{nom}}, \|x\|_{\infty} \leq 1
\]

where \( H = H^{\text{nom}} + \sum_{i=1}^{k} x_k H_i \)

- is this problem convex?
  yes, minimum eigenvalue is a concave function with an affine combination of problem variables
2011 question 1

formulation

\[
\begin{align*}
\text{minimize} & \quad P_n \\
\text{subject to} & \quad P_i = \max_{k \in \text{pred}(i)} \left( P_k \exp(-a_{ik} x_{ik}) \right) \\
& \quad P_1 = 1, \quad 0 \preceq x \preceq x_{\text{max}}, \quad 1^T x \leq B
\end{align*}
\]

- is this problem convex?
  no, equality constraint is not affine, and arguments of maximum are not convex in variables \((x, P)\)

- how do we fix these problem?
  equality constraint can be changed into a inequality constraint, and we can change variables to \(z_i = \log P_i\)