Question 4.1 (Strong duality with conic inequalities): Let $\mathcal{H}$ be a Hilbert space (complete normed vector space with inner product $\langle \cdot, \cdot \rangle$) and $K \subset \mathcal{H}$ be a convex cone with non-empty interior. We say $x \succeq 0$ if $x \in K$ and $x \succ 0$ if $x \in \text{int} K$.

The dual cone associated with $K$ is then $K^* := \{ v \mid \langle v, x \rangle \geq 0 \text{ for all } x \in K \}$.

Let $X$ also be a Hilbert space. A mapping $G : X \to H$ is $K$-convex if its domain is convex and $G(t x + (1 - t) y) \succeq t G(x) + (1 - t) G(y)$ for all $t \in [0, 1]$, $x, y \in \text{dom } G$.

Let $H : \mathcal{X} \to \mathbb{R}^k$ be an affine function. Consider the convex problem

$$\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad G(x) \preceq 0, \quad H(x) = 0, \quad x \in \Omega
\end{align*}$$

where $\Omega \subset \mathcal{X}$ is convex and both $\Omega \subset \text{dom } f$ and $\Omega \subset \text{dom } G$, with optimal value $\omega^\ast$.

Assume the following constraint qualification: there exists $x_0 \in \Omega$ such that $G(x_0) \prec 0$ and $H(x_0) = 0$, (SLATER)

and additionally that $0 \in \text{int} \{ y \in \mathbb{R}^k \mid H(x) = y \text{ for some } x \in \Omega \}$.

Show that strong duality obtains for problem (4.1), that is, there exist $\lambda^\ast \in K^*$ and $\nu^\ast \in \mathbb{R}^k$ such that

$$\inf_{x \in \Omega} \{ f(x) + \langle \lambda^\ast, G(x) \rangle + \langle \nu^\ast, H(x) \rangle \} = \omega^\ast.$$ 

Hints. You may use the following form of the separating hyperplane theorem.

Proposition 4.1.1 (Eidelheit Separation). Let $A, B$ be convex sets in a Hilbert space $\mathcal{X}$ such that $A$ has non-empty interior and $\text{int } A \cap B = \emptyset$. Then there exists $\lambda \in \mathcal{X}$, $\lambda \neq 0$, such that

$$\inf_{a \in A} \langle \lambda, a \rangle \geq \sup_{b \in B} \langle \lambda, b \rangle.$$ 

It will be useful to prove that $A := \{(u, y, t) \mid f(x) \leq t, G(x) \leq u, H(x) = y \text{ for some } x \in \Omega \}$ has non-empty interior. This is not a completely trivial statement. You should feel free to assume that the Hilbert spaces are finite-dimensional (i.e., $\mathbb{R}^n$). I have two solutions; each uses one of the Lemmas 4.1.2 or 4.1.3, both of which follow by combining Proposition 4.1.1 with the following openness guarantee for convex sets.

Lemma 4.1.1 (Hiriart-Urruty and Lemaréchal [1], Lemma III.2.1.6). Let $C$ be a convex set in a vector space with norm $\| \cdot \|$. If $x \in \text{int } C$ and $y \in \text{cl } C$, then the half-open segment $[x, y) := \{(1 - t)x + ty \mid 0 \leq t < 1 \} \subset \text{int } C$. 

---

1For this question, we say that $K$ is a cone if for any $x \in K$, $tx \in K$ for $t > 0$. We do not require that $0 \in K$. 

The book provides the result for $C \subset \mathbb{R}^n$, but the proof extends to any vector space.

If you use one of the following results, you should prove it.

**Lemma 4.1.2.** Let $K$ be a convex cone in a Hilbert space $\mathcal{H}$ with non-empty interior. Then the following hold.

i. Let $u \in \text{int } K$. For $\lambda \in K^*$, $\langle \lambda, u \rangle = 0$ if and only if $\lambda = 0$. In particular, any vector $\lambda \in K^* \cap -K^*$ is zero.

ii. $\text{int } K$ is a convex cone.

iii. For any two vectors $x_0, x_1 \in \mathcal{H}$, $(x_0 + \text{int } K) \cap (x_1 + \text{int } K)$ is non-empty.

**Lemma 4.1.3.** Let $K$ be a convex cone in a Hilbert space $\mathcal{H}$. Then the following hold.

i. Let $u \in \text{int } K$. Then for $\lambda \in K^*$, $\langle \lambda, u \rangle = 0$ if and only if $\lambda = 0$.

ii. $\text{cl } K = K^{**}$, and so if $K$ is closed, then $K = K^{**}$.

iii. Define $\gamma_{\min}(u) := \inf_{\lambda \in K^*} \{ \langle \lambda, u \rangle | \| \lambda \| = 1 \}$. Then $\gamma_{\min}(u) \geq 0$ if and only if $u \in \text{cl } K$. If $\text{int } K \neq \emptyset$, then $\gamma_{\min}(u) > 0$ if and only if $u \succ 0$.

**References**