Problem Session 3

Topics

- Image Filtering
  - Spatial domain vs. Fourier domain
  - Low pass and high pass
- Deconvolution and Inverse Filtering
  - Standard
  - Wiener Deconvolution
- Gradient Descent

Task 1: Image filtering

Primal domain vs. Fourier domain

- Primal: $I(x, y) \rightarrow I(x, y) * PSF(x, y)$

  Point spread function

- Fourier domain: $\tilde{I}(\omega_x, \omega_y) \rightarrow \tilde{I}(\omega_x, \omega_y) \times OTF(\omega_x, \omega_y)$

  Optical transfer function

Task 1: Image Filtering

- Helpful functions: `scipy.signal.convolve2d`, `pypher.psf2otf`, `numpy.fft.fft2`, `numpy.fft.ifft2`
- Normalize the filter so it sums to 1
- You can implement high pass filtering as:

$$I - I * PSF_{LP}$$

Primal Domain

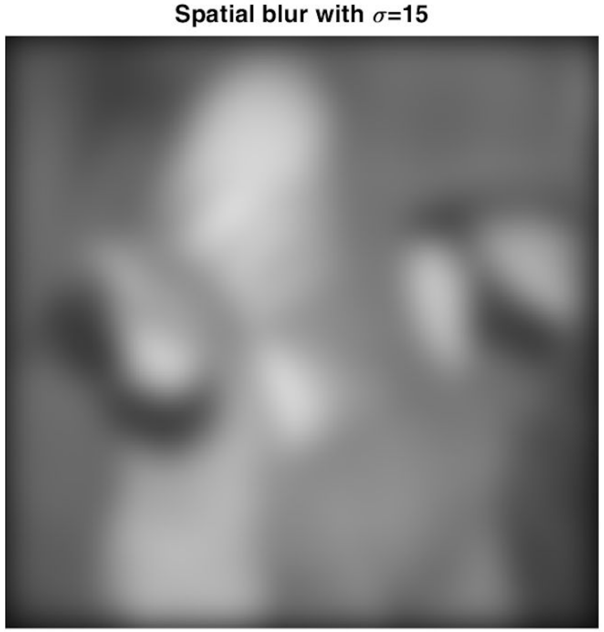
$$\tilde{I} \times (1 - OTF_{LP})$$

Fourier Domain

Task 1: Image filtering

Example of results:

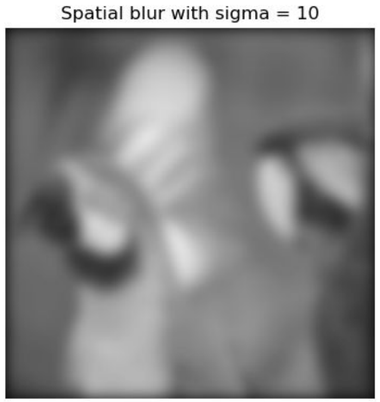
- Primal and dual results look similar

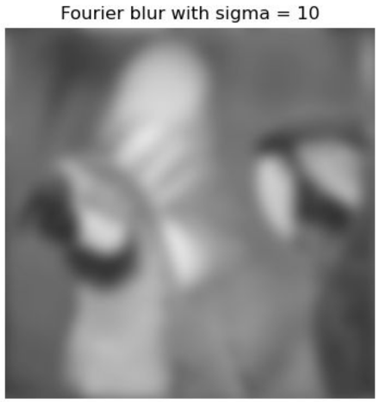


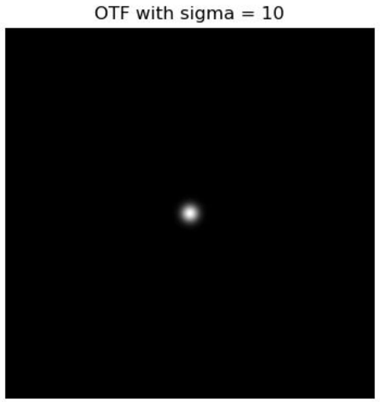
Task 1: Image filtering

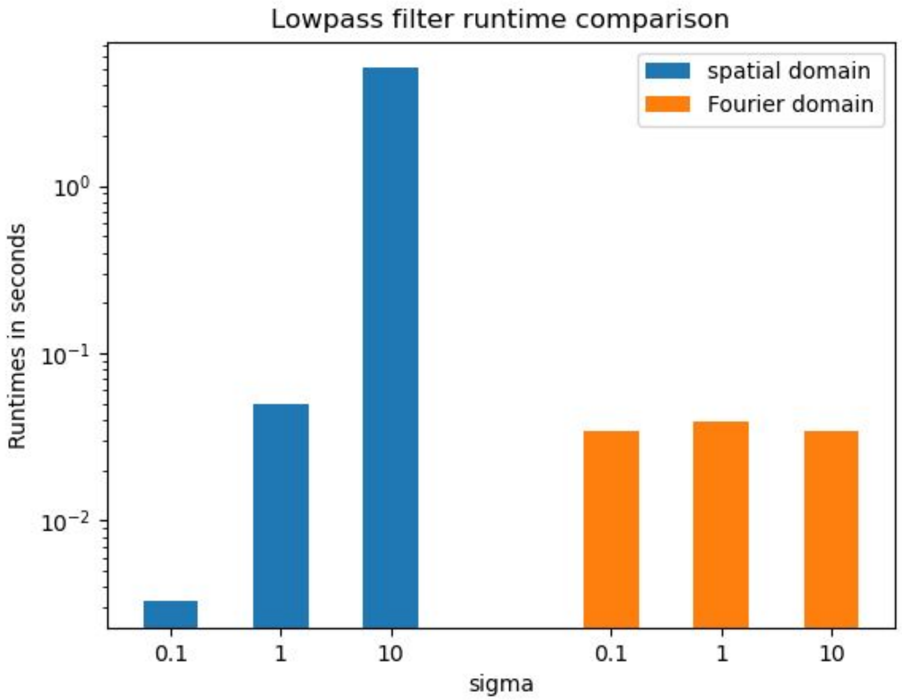
Why are we seeing this behavior?

Example of results

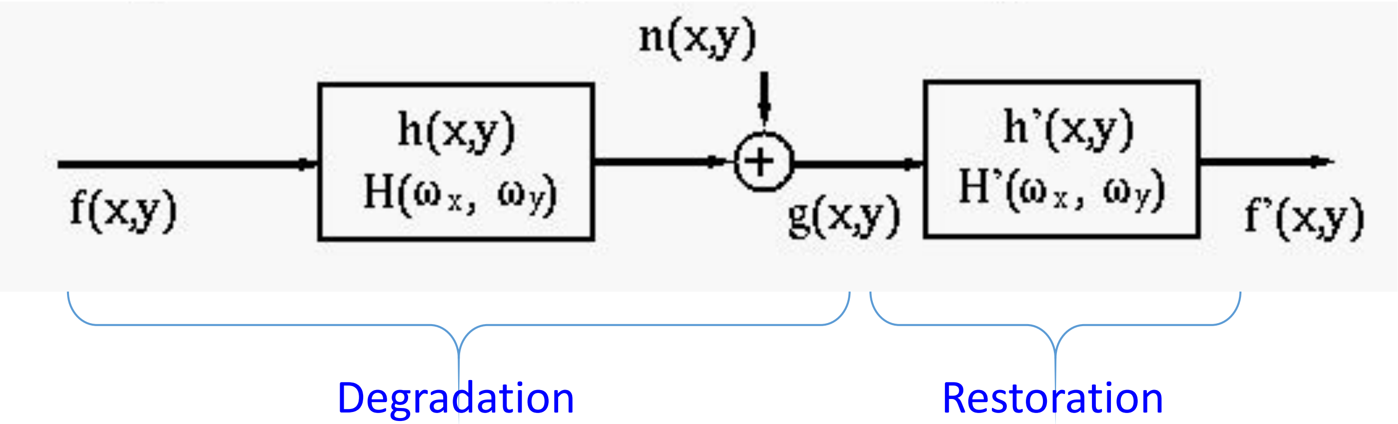








Task 2: Deconvolution and Inverse Filtering



What is the best h' (or H')?

Simply using $H' = 1/H$ will (usually) amplify noise and destroy the image.

Task 2: Deconvolution and Inverse Filtering

For HW:

- First, blur the image with a Gaussian kernel (primal or Fourier domain)
- Add random noise
- Reconstruct the image by
  1. Dividing by the blur kernel (OTF) in Fourier domain (simple inverse filtering)
  2. Wiener deconvolution, which is almost the same as inverse filtering, but uses a damping factor in the Fourier domain that depends on the noise

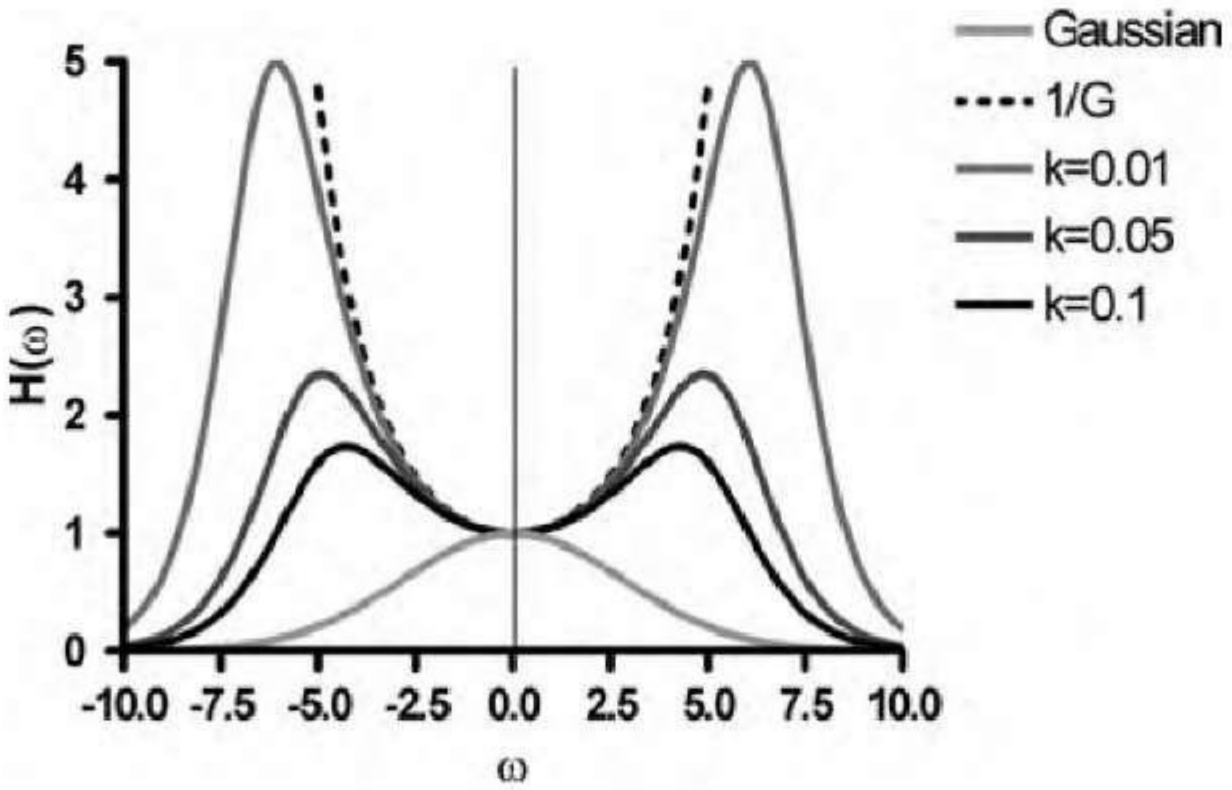
$$H' = \frac{1}{H} \cdot \frac{|H|^2}{(|H|^2 + 1/SNR)} \qquad SNR = \frac{\bar{I}}{\sigma_{noise}}$$

Average pixel value of noisy image

Task 2: Deconvolution and Inverse Filtering

Frequency response of a Wiener filter

$$G' = \frac{1}{G} \cdot \frac{|G|^2}{(|G|^2 + k)}$$



Higher Noise → Lower SNR → More damping → less noise amplification

Task 2: Deconvolution and Inverse Filtering

- The Wiener filter maximizes the probability of the reconstructed signal (S) given the observations (Y).

$$p(S|Y) \propto p(Y|S)p(S)$$

$$\log p(Y|S) = -\|H \odot S - Y\|^2 \qquad \log p(S) = \frac{-1}{SNR}\|S\|^2$$

Task 2: Deconvolution and Inverse Filtering

Calculate the mean squared error (MSE) and the peak signal-to-noise ratio (PSNR):

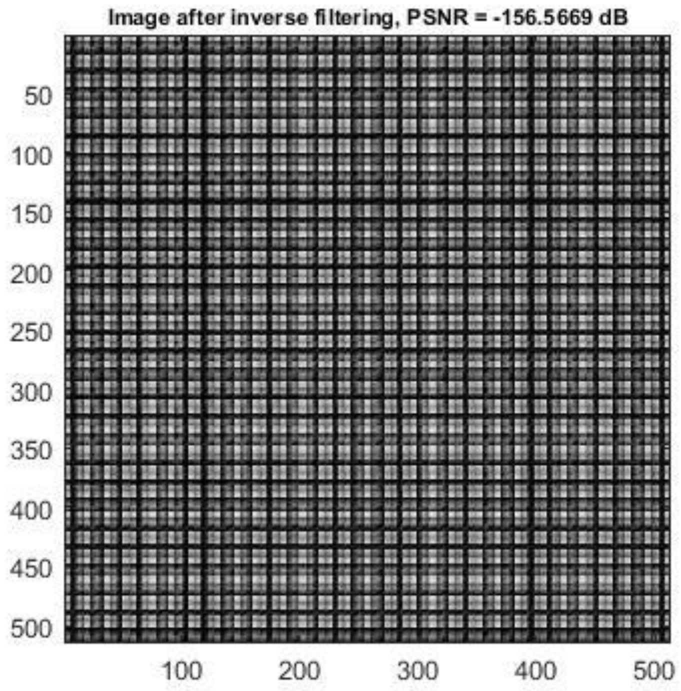
$$MSE = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[I_{original}(i,j) - I_{restored}(i,j)\right]^2$$

$$PSNR = 10\log_{10}\left(\frac{\max(I_{original})^2}{MSE}\right)$$

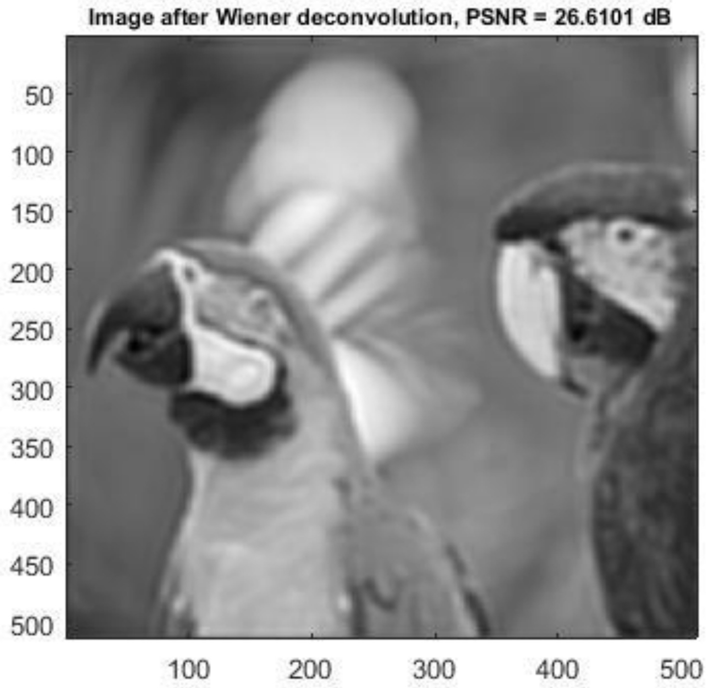
Task 2: Deconvolution and Inverse Filtering

Example of results

Inverse filtering



Wiener deconvolution



Task 3: Gradient Descent

- A general algorithm for solving an optimization problem of the form

$$\underset{x}{\text{minimize}} \quad f(x)$$

- Idea: Move in the direction of the **negative gradient**
  - the direction in which the function is most steeply decreasing
  - Alpha (α) is the step size (or the "learning rate")

$$x^{(k+1)} \leftarrow x^{(k)} - \alpha \nabla f(x^{(k)})$$

Task 3: Gradient Descent

- Apply to the equation

$$\underset{x}{\text{minimize}} \quad \frac{1}{2}\|Ax - b\|_2^2$$

- A: Linear operator representing the image formation model (or **forward model**)
- b: Observed measurements (noisy image)
- x: Desired reconstruction variable

Task 3: Gradient Descent

Residual $$\frac{1}{2}\|Ax - b\|_2^2 = \frac{1}{2}x^TA^TAx - x^TA^Tb + \frac{1}{2}b^Tb$$

$$\nabla_x \left[\frac{1}{2}\|Ax - b\|_2^2\right] = A^TAx - A^Tb$$ **Gradient**

```
def grad_l2(A, x, b):
    # TODO: return the gradient of 0.5 * ||Ax - b||_2^2
    return None


def residual_l2(A, x, b):
    return 0.5 * np.linalg.norm(A @ x - b)**2
```

@ = matrix multiply

Vanilla Gradient Descent

$$x^{(k+1)} \leftarrow x^{(k)} - \alpha \nabla f(x^{(k)})$$

Task 3: Stochastic Gradient Descent

- General case: $x^{(k+1)} \leftarrow x^{(k)} - \alpha g(x^{(k)})$ $\qquad \mathbb{E}[g(x)] = \nabla f(x)$
- In the context of least squares, can express the objective as a sum of scalar residuals:

$$\|Ax - b\|_2^2 = \sum_{i=1}^{n} (a_i^T x - b_i)^2$$

- Choosing a **subset of rows** of A and b === descending on a **subset of these residuals**
  - The number of rows is the **batch size**.
- Use `np.random.randint` to select random indices for the A matrix

Task 3: Gradient Descent

Pass functions as arguments

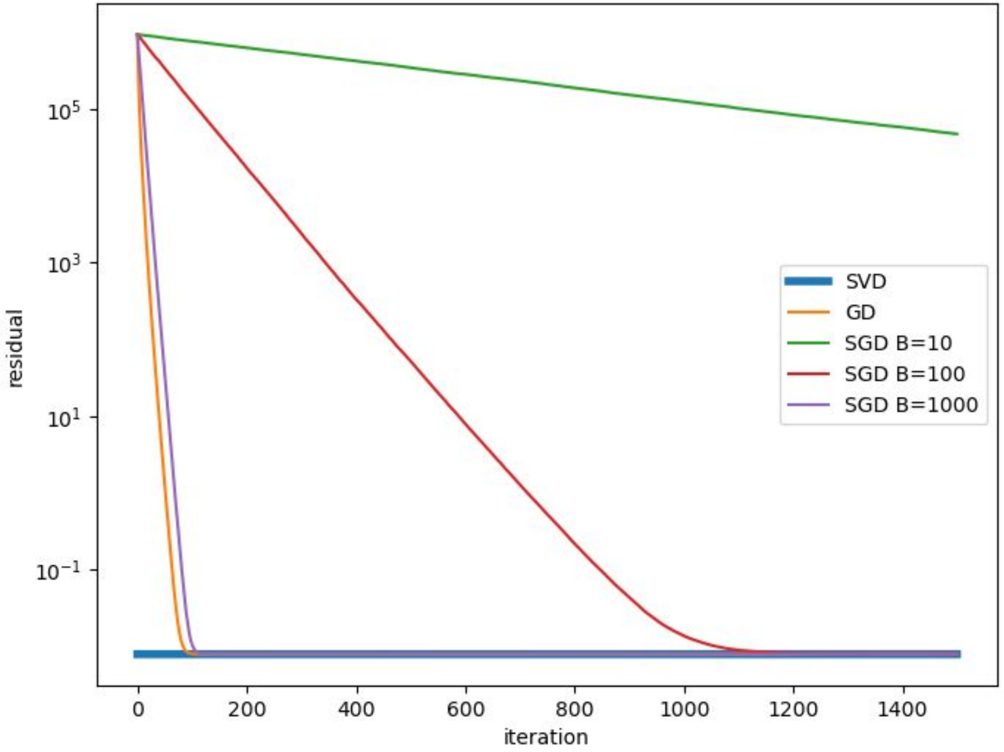
- Functions can be passed as arguments to other functions
- Multiple return values with: `return a, b`
- Unpacking with:

  `a, b = func()`
- `time.time()`

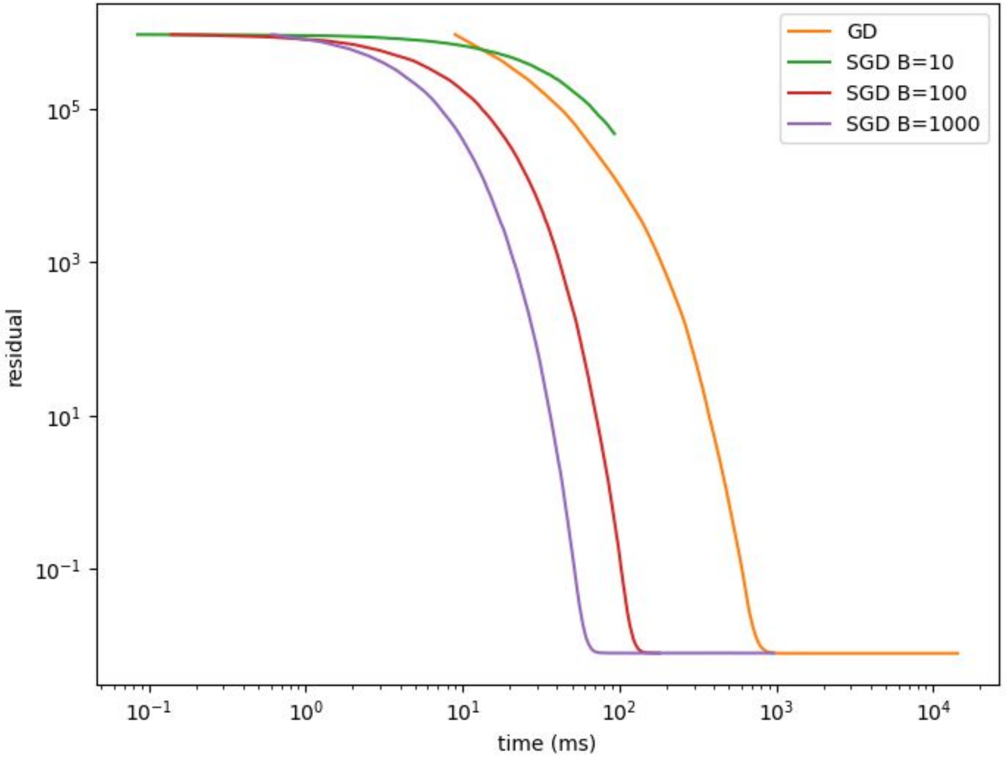
```
def run_gd(A, b, step_size=1e-4, num_iters=1500, grad_fn=grad_l2, residual=residual_l2):
    ''' Run gradient descent to solve Ax = b

    Parameters
    ----------
    A : matrix of size (N_measurements, N_dim)
    b : observations of (N_measurements, 1)
    step_size : gradient descent step size
    num_iters : number of iterations of gradient descent
    grad_fn : function to compute the gradient
    residual : function to compute the residual

    Returns
    -------
    x
        output matrix of size (N_dim)

    residual
        list of calculated residuals at each iteration

    timing
        time to execute each iteration (should be cumulative to each iteration)

    '''
```

Multiple return values

Task 3: Gradient Descent



Use full A matrix, not subsampled A, to compute residual.



Note: Exact runtimes and order of convergence in wall clock time may vary!

Have a nice weekend!

And good luck with the homework!