Eigenimages

- Unitary transforms
- Karhunen-Loève transform and eigenimages
- Sirovich and Kirby method
- Eigenfaces for gender recognition
- Fisher linear discriminant analysis
- Fisherimages and varying illumination
- Fisherfaces vs. eigenfaces
To recognize complex patterns (e.g., faces), large portions of an image (say $N$ pixels) have to be considered.

High dimensionality of “image space” results in high computational burden for many recognition techniques.  

Example: nearest-neighbor search requires pairwise comparison with every image in a database.

Transform $\vec{c} = W\vec{f}$ is a projection on a $J$-dimensional linear subspace that greatly reduces the dimensionality of the image space $J << N$.

Idea: tailor the projection to a set of representative training images and preserve the salient features by using Principal Component Analysis (PCA).

$$W_{\text{opt}} = \arg \max_W \det(WR_{ff}W^H)$$

Mean squared value of projection

JxN projection matrix with orthonormal rows.

Autocorrelation matrix of image
Image recognition using linear projection

2-d example:

Goal: project samples on a 1-d subspace, then perform classification.
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Unitary transforms

- Sort pixels $f[x,y]$ of an image into column vector $\vec{f}$ of length $N$
- Calculate $N$ transform coefficients

\[ \vec{c} = A \vec{f} \]

where $A$ is a matrix of size $N \times N$

- The transform $A$ is unitary, iff

\[ A^{-1} = A^{*T} \equiv A^H \]

Hermitian conjugate

- If $A$ is real-valued, i.e., $A = A^*$, transform is „orthonormal“
Energy conservation with unitary transforms

- For any unitary transform \( \vec{c} = A\vec{f} \) we obtain

\[
\|\vec{c}\|^2 = \vec{c}^H\vec{c} = \vec{f}^H A^H A\vec{f} = \|\vec{f}\|^2
\]

- Interpretation: every unitary transform is simply a rotation of the coordinate system (and, possibly, sign flips)
- Vector length is conserved.
- Energy (mean squared vector length) is conserved.
Energy distribution for unitary transforms

- Energy is conserved, but, in general, unevenly distributed among coefficients.
- Autocorrelation matrix

\[ R_{cc} = E \left[ \overline{c}c^H \right] = E \left[ A \overrightarrow{f} \cdot \overrightarrow{f}^H A^H \right] = AR_{ff}A^H \]

- Diagonal of \( R_{cc} \) comprises mean squared values ("energies") of the coefficients \( c_i \)

\[ E \left[ c_i^2 \right] = \left[ R_{cc} \right]_{i,i} = \left[ AR_{ff}A^H \right]_{i,i} \]
Eigenmatrix of the autocorrelation matrix

**Definition:** eigenmatrix $\Phi$ of autocorrelation matrix $R_{ff}$

- $\Phi$ is unitary
- The columns of $\Phi$ form a set of eigenvectors of $R_{ff}$, i.e.,

$$R_{ff} \Phi = \Phi \Lambda$$

$\Lambda$ is a diagonal matrix of eigenvalues $\lambda_i$

$$\Lambda = \begin{pmatrix}
\lambda_0 & 0 \\
0 & \lambda_1 \\
\vdots & \vdots \\
0 & \lambda_{N-1}
\end{pmatrix}$$

- $R_{ff}$ is normal matrix, i.e., $R_{ff}^H R_{ff} = R_{ff} R_{ff}^H$, hence unitary eigenmatrix exists ("spectral theorem"")
- $R_{ff}$ is symmetric nonnegative definite, hence $\lambda_i \geq 0$ for all $i$
Karhunen-Loève transform

- Unitary transform with matrix

\[ A = \Phi^H \]

- Transform coefficients are pairwise uncorrelated

\[ R_{cc} = A R_{ff} A^H = \Phi^H R_{ff} \Phi = \Phi^H \Phi \Lambda = \Lambda \]

- Columns of \( \Phi \) are ordered according to decreasing eigenvalues.

- Energy concentration property:
  - No other unitary transform packs as much energy into the first \( J \) coefficients.
  - Mean squared approximation error by keeping only first \( J \) coefficients is minimized.
  - Holds for any \( J \).
Illustration of energy concentration

Strongly correlated samples, equal energies

After KLT: uncorrelated samples, most of the energy in first coefficient

\[ \mathbf{A} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]
Basis images and eigenimages

For any transform, the inverse transform

$$\vec{f} = A^{-1}\vec{c}$$

can be interpreted in terms of the superposition of columns of $A^{-1}$ ("basis images")

For the KL transform, the basis images are the eigenvectors of the autocorrelation matrix $R_{ff}$ and are called "eigenimages."

If energy concentration works well, only a limited number of eigenimages is needed to approximate a set of images with small error. These eigenimages span an optimal linear subspace of dimensionality $J$.

Eigenimages can be used directly as rows of the projection matrix

$$W_{opt} = \arg \max_W \det(WR_{ff}W^H)$$

Mean squared value of projection

JxN projection matrix with orthonormal rows

Autocorrelation matrix of image
Eigenimages for face recognition

New Face Image

Normalization

\[ \frac{f}{\|f\|} \]

Projection

\[ W \]

\[ c \]

Database of Eigenface Coefficients

\[ p_1, \ldots, p_k \]

Similarity Matching

Class of most similar \( p_k \)

Similarity measure (e.g., \( c^T \hat{p}_k \))

Recognition Result

Rejection

1

\( k^* \)

Recognition Result
Computing eigenimages from a training set

- How to obtain \(N \times N\) covariance matrix?
  - Use training set \(\bar{\Gamma}_1, \bar{\Gamma}_2, \ldots, \bar{\Gamma}_{L+1}\) (each column vector represents one image)
  - Let \(\mu\) be the mean image of all \(L+1\) training images
  - Define training set matrix 
    \[
    S = \left( \bar{\Gamma}_1 - \mu, \bar{\Gamma}_2 - \mu, \bar{\Gamma}_3 - \mu, \ldots, \bar{\Gamma}_L - \mu \right),
    \]
    and calculate scatter matrix 
    \[
    R = \sum_{l=1}^{L} \left( \bar{\Gamma}_l - \mu \right) \left( \bar{\Gamma}_l - \mu \right)^H = SS^H
    \]

  Problem 1: Training set size should be \(L + 1 >> N\)
  
  If \(L < N\), scatter matrix \(R\) is rank-deficient

  Problem 2: Finding eigenvectors of an \(N \times N\) matrix.

- Can we find a small set of the most important eigenimages from a small training set \(L << N\) ?
Sirovich and Kirby algorithm

- Instead of eigenvectors of $SS^H$, consider the eigenvectors of $S^H S$, i.e.,
  \[ S^H S \vec{v}_i = \lambda_i \vec{v}_i \]
- Premultiply both sides by $S$
  \[ SS^H S \vec{v}_i = \lambda_i S \vec{v}_i \]
- By inspection, we find that $S \vec{v}_i$ are eigenvectors of $SS^H$

Sirovich and Kirby Algorithm (for $L << N$)

- Compute the $L \times L$ matrix $S^H S$
- Compute $L$ eigenvectors $\vec{v}_i$ of $S^H S$
- Compute eigenimages corresponding to the $L_0 \leq L$ largest eigenvalues as a linear combination of training images $S \vec{v}_i$

Example: eigenfaces

- The first 8 eigenfaces obtained from a training set of 100 male and 100 female training images

  Mean Face

  Eigenface 1

  Eigenface 2

  Eigenface 3

  Eigenface 4

  Eigenface 5

  Eigenface 6

  Eigenface 7

  Eigenface 8

- Can be used to generate faces by adjusting 8 coefficients.
- Can be used for face recognition by nearest-neighbor search in 8-d „face space.“
Gender recognition using eigenfaces

Nearest neighbor search in “face space”

Recognition Rate

Number of Eigenfaces

Female face samples

Male face samples

Gender recognition using eigenfaces
Fisher linear discriminant analysis

- Eigenimage method maximizes “scatter” within the linear subspace over the entire image set – regardless of classification task

\[ W_{opt} = \arg \max_{W} \left( \det(WRW^H) \right) \]

- Fisher linear discriminant analysis (1936): maximize between-class scatter, while minimizing within-class scatter

\[ R_B = \sum_{i=1}^{c} N_i \left( \mu_i - \bar{\mu} \right) \left( \mu_i - \bar{\mu} \right)^H \]

\[ R_W = \sum_{i=1}^{c} \sum_{\Gamma_i \in \text{Class}(i)} \left( \Gamma_i - \bar{\mu}_i \right) \left( \Gamma_i - \bar{\mu}_i \right)^H \]
Fisher linear discriminant analysis (cont.)

- Solution: Generalized eigenvectors $\overrightarrow{w}_i$ corresponding to the $J$ largest eigenvalues $\{\lambda_i | i = 1, 2, ..., J\}$, i.e.

$$R_B \overrightarrow{w}_i = \lambda_i R_W \overrightarrow{w}_i, \quad i = 1, 2, ..., J$$

- Problem: within-class scatter matrix $R_w$ at most of rank $L-c$, hence usually singular.

- Apply KLT first to reduce dimensionality of feature space to $L-c$ (or less), proceed with Fisher LDA in lower-dimensional space
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Eigenimages vs. Fisherimages

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Fisher LDA separates the classes by choosing a better 1-d subspace.
Fisher images and varying illumination

Differences due to varying illumination can be much larger than differences among faces!
Fisherimages and varying illumination

- All images of same Lambertian surface with different illumination (without shadows) lie in a 3d linear subspace
- Single point source at infinity

\[
f(x,y) = a(x,y)(\vec{l}^T\vec{n}(x,y))L
\]

- Superposition of arbitrary number of point sources at infinity still in same 3d linear subspace, due to linear superposition of each contribution to image
- Fisherimages can eliminate within-class scatter
Fisherface trained to recognize gender

Female face samples

Male face samples

Mean image $\bar{\mu}$
Female mean $\bar{\mu}_1$
Male mean $\bar{\mu}_2$

Fisherface
Gender recognition using 1\textsuperscript{st} Fisherface

Error rate = 6.5%
Gender recognition using 1\textsuperscript{st} eigenface

Error rate = 19.0%
Person identification with Fisherfaces and eigenfaces

ATT Database of Faces
- 40 classes
- 10 images per class