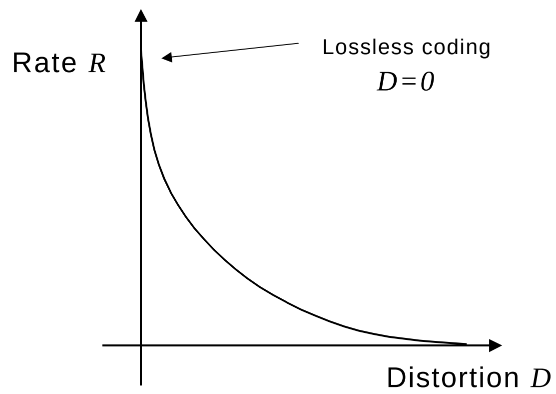
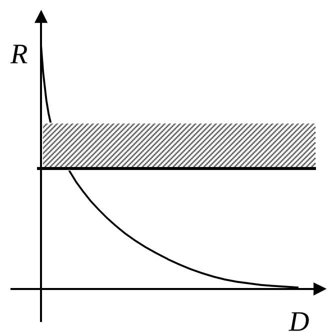
Lossy compression

- Lower the bit-rate R by allowing some acceptable distortion D of the signal.

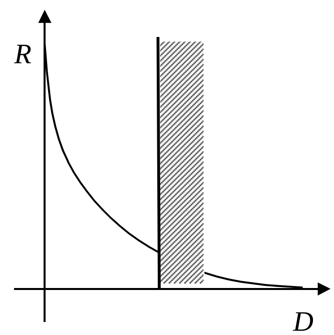


Types of lossy compression problems

- Given maximum rate R, minimize distortion D



- Given distortion D, minimize rate R

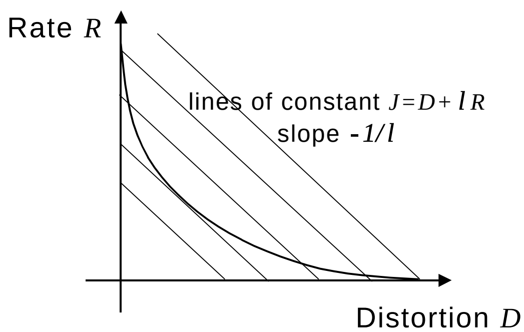


Equivalent constrained optimization problems, often unwieldy due to constraint.

Lagrangian formulation of the R-D problem

- Instead of cost function D, with constrained R, or R, with constrained D, use <u>unconstrained</u> Lagrangian cost function

$$J = D + \lambda R$$



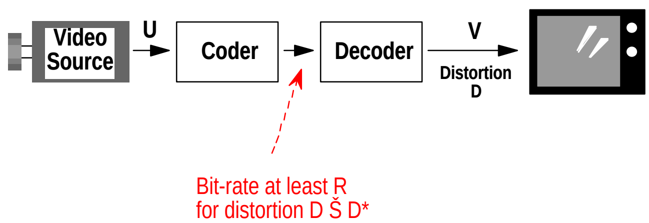
- Requires convex R-D function

Topics in lossy compression

- Information theoretical bounds for lossy compression: rate distortion theory
  - R-D function for memoryless Gaussian sources with mean-squared error distortion criterion
  - R-D function for Gaussian sources with memory
  - R-D for images
- Practical lossy compression techniques: quantization
  - Scalar quantization and vector quantization
  - Quantizer design for fixed length codes
  - Quantizer design for variable length codes

Rate distortion theory

- Rate distortion theory calculates the minimum transmission bit-rate R for a required picture quality.



- Results of rate distortion theory are obtained without consideration of a specific coding method.

Mutual information

- "Mutual information" is the information that symbols *u* and symbols *v* convey about each other.
- Average mutual information:

$$\boxed{\begin{aligned} I(U;V) &= H(V) - H(V \mid U) = H(U) - H(U \mid V) \\ &= \sum_u \sum_v P(u,v) \log \frac{P(u,v)}{P(u)P(v)} \end{aligned}}$$

- Properties of mutual information:

$$\boxed{\begin{gathered} 0 \le I(U;V) = I(V;U) \\ I(U;V) \le H(U) \\ I(V;U) \le H(V) \end{gathered}}$$

- **Channel coding:** channel capacity *C* is maximum of mutual information between transmitter and receiver.

Distortion

- Symbol *u* sent, *v* received
- Per-symbol distortion:

$$\begin{array}{l} d(u,v) \geq 0 \\ d(u,v) = 0 \ \text{ for } u = v \end{array}$$

- Average distortion:

$$\boxed{D(U,V) = E\{d(U,V)\} = \sum_u \sum_v P(u,v)d(u,v)}$$

- Distortion criterion:

$$\boxed{D \leq D^*}$$

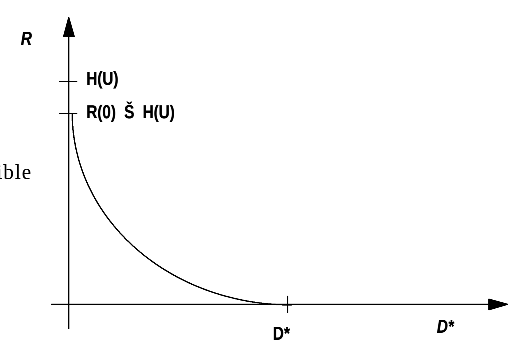
maximum average distortion

Rate distortion function

- Definition:

$$\boxed{R(D^*) = \min_{D \leq D^*} \{I(U;V)\}}$$

- For a given maximum average distortion *D**, the rate distortion function *R(D*)* is the lower bound for the transmission bit-rate.
- Typical *R(D*)* function:



$H(U)$: entropy of the source

$R(0)$: rate with just not perceptible loss of quality

Shannon Lower Bound

- It can be shown that $H(U-V\,|\,V) = H(U\,|\,V)$
- Thus

$$
\begin{aligned}
R(D^*) &= \min_{D \le D^*} \{ H(U) - H(U\,|\,V) \} \\
&= H(U) - \max_{D \le D^*} \{ H(U\,|\,V) \} \\
&= H(U) - \max_{D \le D^*} \{ H(U-V\,|\,V) \}
\end{aligned}
$$

- Ideally, the source coder would produce distortions *u*-*v* that are statistically independent from the reconstructed signal *v* (not always possible!).
- Shannon lower bound:

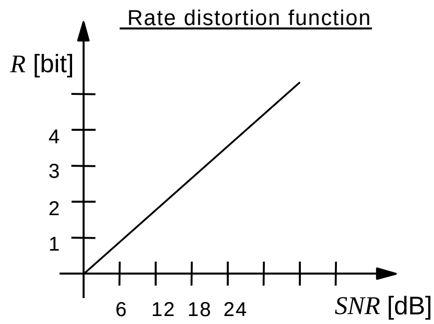
$$\boxed{R(D^*) \ge H(U) - \max_{D \le D^*} H(U-V)}$$

R(D)* function for a memoryless Gaussian source and MSE distortion

- Gaussian source, variance s^2
- Mean squared error

$$D = E\{(U-V)^2\}$$

- Rule of thumb: 6 dB ≅ 1 bit
- The *R(D*)* for non-Gaussian sources with the same variance s^2 is always <u>below</u> this Gaussian *R(D*)*-curve.



$$\boxed{R(D^*) = \frac{1}{2}\log\left(\frac{s^2}{D^*}\right)}$$

$$SNR = 10\log_{10}\left(\frac{s^2}{D^*}\right) \text{ [dB]}$$

R(D)* function for Gaussian source with memory I

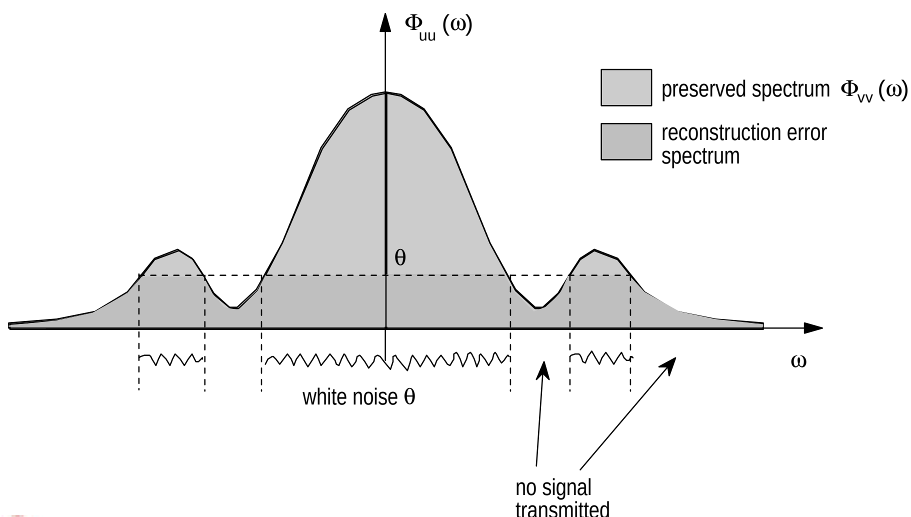
- Jointly Gaussian source with power spectrum $\Phi_{uu}(w)$
- Mean squared error distortion $D = E\{(U-V)^2\}$
- Parametric formulation of the *R(D*)* function

$$D(q) = \frac{1}{2p}\int_w \min\{q, \Phi_{uu}(w)\} dw$$

$$R(q) = \frac{1}{2p}\int_w \max\left\{0, \frac{1}{2}\log\frac{\Phi_{uu}(w)}{q}\right\} dw$$

- *R(D*)* for non-Gaussian sources with the same power spectral density is always lower.

R(D)* function for Gaussian source with memory II



Rate distortion function for images I

- Signal model: Gaussian source with acf

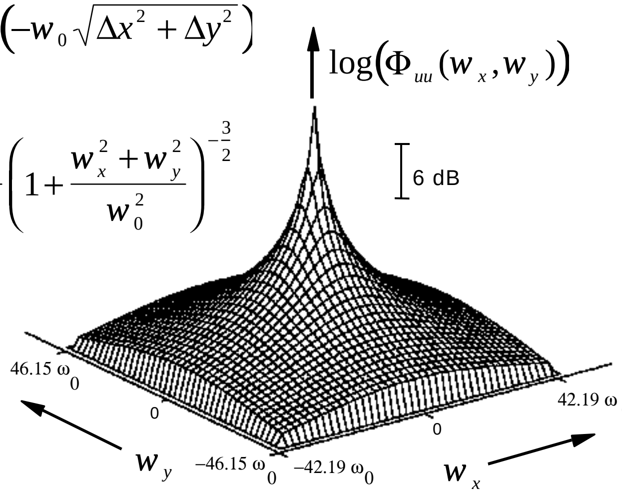
$$R_{uu}(\Delta x, \Delta y) = \exp\left(-w_0 \sqrt{\Delta x^2 + \Delta y^2}\right)$$

- Power spectral d

$$\Phi_{uu}(w_x, w_y) = \frac{2p}{w_0^2}\left(1 + \frac{w_x^2 + w_y^2}{w_0^2}\right)^{-\frac{3}{2}}$$

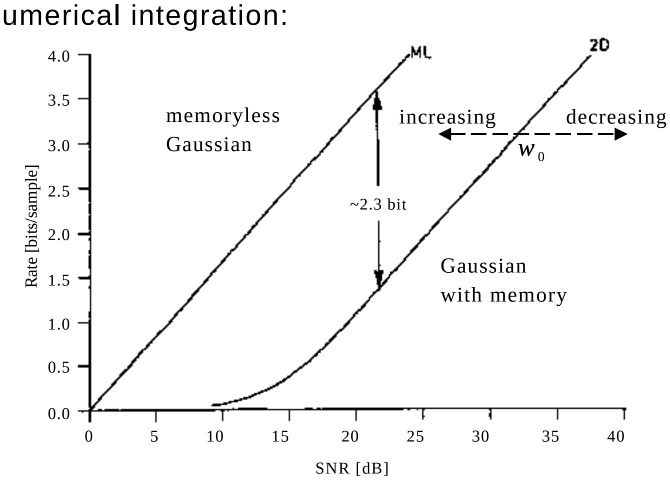
$$w_0 = -\ln(0.93)$$

correlation between adjacent pixels



Rate distortion function for images II

- Mean squared error criterion: $D = E\{(U - V)^2\}$
- After numerical integration:



Summary: rate distortion theory

- Lagrangian cost function $J=D+\lambda R$ can be used instead of solving constrained rate or distortion minimization.
- Rate-distortion theory: minimum transmission bit-rate for given distortion
- Shannon Lower Bound assumes statistical independence between distortion and reconstructed signal
- *R(D*)* for memoryless Gaussian source and MSE: 6 dB/bit
- *R(D*)* for Gaussian source with memory and MSE: encode spectral components independently, introduce white noise, suppress small spectral components
- Theoretical gain ~2.3 bits/sample by exploiting spatial redundancy in the video signal