EE376A: Review Session 1 Solutions

Channel Capacity

1 Modulo Channel

(a) Consider the DMC defined as follows: Output $Y = X \oplus_2 Z$ where X, taking values in $\{0, 1\}$, is the channel input, $\oplus_2$ is the modulo-2 summation operation, and Z is binary channel noise uniformly distributed over $\{0, 1\}$ and independent of X. What is the capacity of this channel?

Solution
The channel is a BSC with crossover probability 1/2, so the capacity is zero.

(b) Consider the channel of the previous part, but suppose that instead of modulo-2 addition $Y = X \oplus_2 Z$, we perform modulo-3 addition $Y = X \oplus_3 Z$. Now what is the capacity?

Solution
Note that now $Y \in \{0, 1, 2\}$. Since Z is Bernoulli($\frac{1}{2}$), the channel output being 1 indicates that X could be 0 or 1 with equal probability. Thus, it gives no information about the input. This simply becomes a Binary Erasure channel with erasure probability $\frac{1}{2}$, and the capacity is $1 - \frac{1}{2} = \frac{1}{2}$.

(c) Now suppose the noise Z is no longer independent of the input X, but is instead described by the following conditional distribution:

$$p(Z = z|X = 0) = \begin{cases} 1/4 \text{ if } z = 0 \\ 3/4 \text{ if } z = 1, \end{cases}$$

and

$$p(Z = z|X = 1) = 1/2 \quad \text{both for } z = 0 \text{ and } z = 1.$$

A random code of size 2^{nR} is generated uniformly (that is all codewords are drawn i.i.d. $X \sim Bern(0.5)$). Find the value V such that if $R < V$ then the average probability of decoding error (average both across the messages and the randomness in the codebook) vanishes with increasing blocklength while if $R > V$ then it does not.

Solution
Recall from the direct and converse theorems in the lecture that the maximum supported rate under a given random codebook is simply the Mutual Information between X and Y.
If X is uniform and the noise Z is distributed as described, one may determine the information between X and Y as follows:

$$I(X; Y) = H(Y) - H(Y|X) = H(Y) - H(Z|X).$$

For X chosen Bernoulli(1/2), we have that

$$H(Z|X) = \frac{1}{2} h_b\left(\frac{1}{4}\right) + \frac{1}{2}.$$

To compute $H(Y)$, we first observe that the distribution of Y when X is Bernoulli(1/2) is given by $(p_0, p_1, p_2) = (\frac{1}{8}, \frac{5}{8}, \frac{1}{4})$. The entropy of the this distribution may then be calculated.

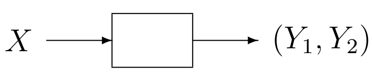
Performing both computations, we find that $I(X;Y) = H(Y) - H(Z|X) = 0.393$. This is the maximum supported rate by the channel under a uniformly generated codebook. Note that this is not the capacity of the channel which is achieved for a different input distribution.

2 A channel with two independent looks at Y.

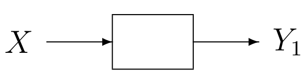
Let Y_1 and Y_2 be conditionally independent and conditionally identically distributed given X. Thus $p(y_1, y_2|x) = p(y_1|x)p(y_2|x)$.

(a) Show $I(X;Y_1,Y_2) = 2I(X;Y_1) - I(Y_1;Y_2)$.

(b) Conclude that the capacity of the channel



is less than twice the capacity of the channel



Solution: A channel with two independent looks at Y.

(a) First note that Y_1 and Y_2 are identically distributed since $p(y_1|x) = p(y_2|x)$ and hence $p(y_1) = \sum_x p(y_1|x)p(x) = \sum_x p(y_2|x)p(x) = p(y_2)$. Therefore,

$$\begin{aligned} I(X;Y_1,Y_2) &= H(Y_1,Y_2) - H(Y_1,Y_2|X) \\ &= H(Y_1) + H(Y_2) - I(Y_1;Y_2) - H(Y_1,Y_2|X) \\ &= H(Y_1) + H(Y_2) - I(Y_1;Y_2) - H(Y_1|X) - H(Y_2|X) \quad (1) \\ &= 2H(Y_1) - 2H(Y_1|X) - I(Y_1;Y_2) \quad (2) \\ &= 2I(X;Y_1) - I(Y_1;Y_2), \end{aligned}$$

where Eq. (1) follows from the fact that Y_1 and Y_2 are conditionally independent given X and Eq. (2) follows from the fact that Y_1 and Y_2 are identically distributed and conditionally identically distributed given X.

(b) The capacity of the single look channel $X \to Y_1$ is

$$C_1 = \max_{p(x)} I(X;Y_1).$$

The capacity of the channel $X \to (Y_1, Y_2)$ is

$$\begin{aligned} C_2 &= \max_{p(x)} I(X; Y_1, Y_2) \\ &= \max_{p(x)} 2I(X; Y_1) - I(Y_1; Y_2) \\ &\leq \max_{p(x)} 2I(X; Y_1) \\ &= 2C_1. \end{aligned}$$

Hence, two independent looks cannot be more than twice as good as one look.