EE376A Information Theory Lecture 19 - 03/12/2015

Lecture 19 - Joint Source-Channel Coding Examples

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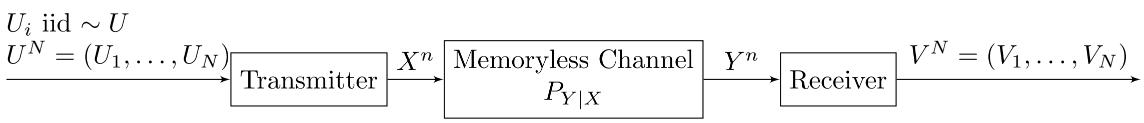
1 Joint Source-Channel Coding

1.1 Recap from last lecture

The goal is to communicate the $U^N = (U_1, U_2, \ldots, U_N)$ through the memoryless channel given by $P_{Y|X}$ with small expected distortion, measured by $\mathbb{E}\left[d(U^N, V^N)\right]$. Note that the U_i are not necessarily bits. A schematic of this setup is shown below:



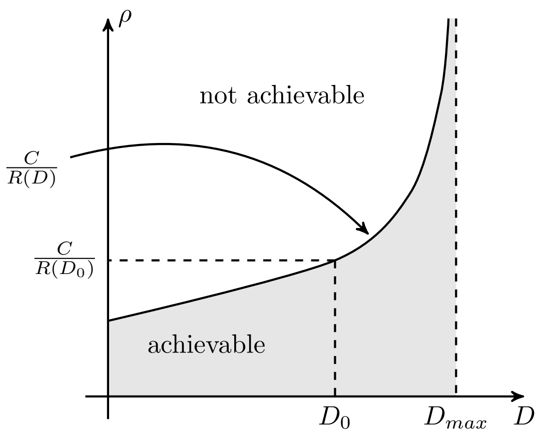
The rate of communication is then

$$\text{rate} = \frac{N}{n} \frac{\text{source symbols}}{\text{channel use}}$$

We also allow an expected distortion $\mathbb{E}\left[d(U^N, V^N)\right]$.

A rate-distortion pair (ρ, D) is achievable if $\forall \epsilon > 0 \; \exists$ scheme with $\frac{N}{n} \geq D + \epsilon$ and $\mathbb{E}[d(U^N, V^N)] \leq D + \epsilon$

Last lecture, we proved the "Source-Channel Separation Theorem":
(ρ, D) is achievable if and only if $\rho R(D) \leq C$



We can achieve points on the curve above by first thinking about representing the data as bits in an efficient manner (compression) with rate $R(D)$ and then transmitting these bits losslessly across the channel with rate C. Note that the distortion without sending anything over the channel is D_{max}.

Example 1. Binary Source and Binary Channel

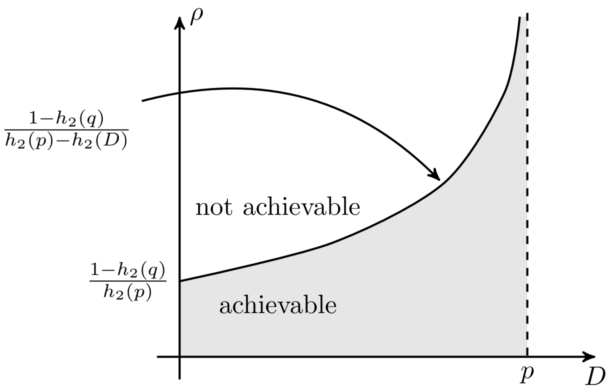
Source: $U \sim \text{Ber}(p), \quad 0 \leq p \leq 1/2$

Channel: BSC(q), $0 \le q \le 1/2$
Distortion: Hamming

Recall that for $U \sim \text{Ber}(p)$, the rate distortion function is $R(D) = h_2(p) - h_2(D)$ and that a binary symmetric channel with crossover probability q has capacity $C = 1 - h_2(q)$

So, we see that if we want distortion $\le D$, then (for $D \le p$) the maximum achievable rate is:

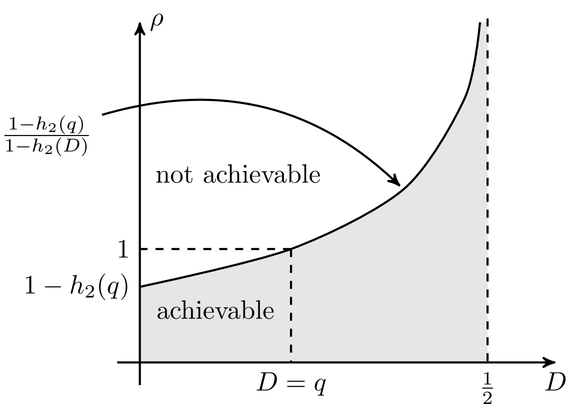
$$\rho = \frac{1 - h_2(q)}{h_2(p) - h_2(D)}$$



Note that the communication problem corresponds to $D = 0$.

In particular, if $p = 1/2$, then if we want distortion $\le D$, the maximum rate we can transmit at is:

$$\rho = \frac{1 - h_2(q)}{1 - h_2(D)}$$



Consider the following scheme for rate=1:

Channel input: $X_i = U_i$
reconstruction: $V_i = Y_i$

The expected distortion is then $P(U_i \neq V_i) = P(X_i \neq Y_i) = q$

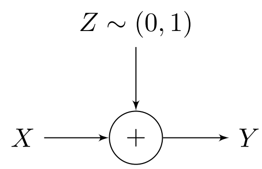
$\longrightarrow$ this scheme is optimal, since $\rho = (1 - h_2(q))/(1 - h_2(D = q)) = 1$.

In this particular case, it is possible to achieve the optimal rate using a scheme that individually encodes and transmits each symbol.

Example 2. Gaussian Source and Gaussian Channel

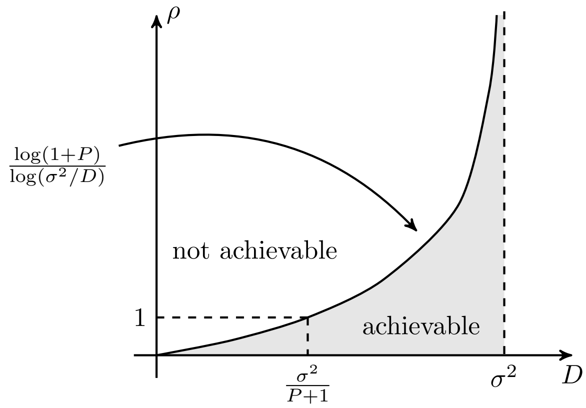
Source: $U \sim \mathcal{N}(1, \sigma^2)$

Channel: AWGN (Additive White Gaussian Noise Channel) with power constraint P



distortion: squared error

Recall that for $U \sim \mathcal{N}(1, \sigma^2)$, the rate distortion function is $R(D) = \frac{1}{2}\log(\frac{\sigma^2}{D})$ (for $0 \le D \le \sigma^2$) and that the AWGN channel with power constraint P has capacity $C = \frac{1}{2}\log(1+P)$



Then, for a given distortion $D \le \sigma^2$, the maximum achievable rate is

$$\rho = \frac{\log(1+P)}{\log(\sigma^2/D)}$$

Consider the following scheme at rate=1:

transmit: $X_i = \sqrt{\frac{P}{\sigma^2}} U_i$

receive: $Y_i = X_i + Z_i = \sqrt{\frac{P}{\sigma^2}} U_i + Z_i$
reconstruction: $V_i = \mathbb{E}[U_i|V_i]$

The distortion is squared error, so we know that reconstruction using the expected value is optimal. Thus, we take $V_i = \mathbb{E}[U_i|V_i]$.

The expected distortion is then:

$$\begin{aligned}
\mathbb{E}[U_i|V_i] &= \mathrm{Var}(U_i|V_i) \\
&= \mathrm{Var}\left(U_i \middle| \sqrt{\frac{\sigma^2}{P}} Y_i\right) \\
&= \mathrm{Var}\left(U_i \middle| U_i + \sqrt{\frac{\sigma^2}{P}} Z_i\right) \\
&\overset{(a)}{=} \frac{\sigma^2(\sigma^2/P)}{\sigma^2 + \sigma^2/P} \\
&= \frac{\sigma^2}{P+1}
\end{aligned}$$

where (a) follows from the fact that for $X \sim \mathcal{N}(0, \sigma_1^2)$ independent from $Y \sim \mathcal{N}(0, \sigma_2^2)$:

$$\mathrm{Var}(X|X+Y) = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

Now, at rate $= 1$:
The optimal D satisfies

$$\begin{aligned}
&\frac{\log(1+P)}{\log(\sigma^2/D)} = 1 \\
&\rightarrow 1 + P = \frac{\sigma^2}{D}
\end{aligned}$$

So, in the specific case of rate $= 1$ we see that the simple scheme above is optimal, just as the simple scheme for the Binary Source and Channel was also optimal when rate $= 1$.