EE 378A Statistical Signal Processing	Professor Andrea Montanari
Homework 1	
Due on: 04/18/2022	

Problem 1

Equivalence between different types of estimators
Adapted from Exercise 1.11 from A. Tsybakov, Introduction to Nonparametric Estimation
Consider the regression model under the following assumptions:

(i) We consider the nonparametric regression model

$$Y_l = f(t_l) + \sigma z_l,\ l \in \{0, \dots, n-1\},$$

where f is a function from $[0,1]$ to $\mathbb{C}$. The random variables z_l are i.i.d. complex Gaussian with $z_l \sim \mathcal{N}_{\mathbb{C}}(0,1)$ and $t_l = l/n$ for $l \in \{0, \dots, n-1\}$.

(ii) $(\phi_q(t))_{q\in\mathbb{Z}}$ is the trigonometric basis:

$$\phi_q(t) = e^{i2\pi qt}$$

(iii) The Fourier coefficients $b_q = \int_0^1 f(t)\phi_q(t)dt$ of f satisfy

$$\sum_{q\in\mathbb{Z}} |b_q| < \infty$$

The smoothing spline estimator $f_n^{\mathrm{sp}}(t)$ is defined as a solution of the following minimization problem:

$$f_n^{\mathrm{sp}} = \operatorname*{argmin}_{f\in W} \left[\frac{1}{n} \sum_{l=0}^{n-1} |Y_l - f(t_l)|^2 + \kappa \int_0^1 |f''(t)|^2 \, dt \right], \tag{1}$$

where $\kappa > 0$ is a smoothing parameter and W is one of the sets of functions defined below.

(a) First suppose that W is the set of all the functions $f : [0,1] \to \mathbb{C}$ such that f' is absolutely continuous. Prove that the estimator f_n^{sp} reproduces polynomials of degree ≤ 1 if $n \ge 2$ (i.e. if $f(t) = \alpha t + \beta$ for some $\alpha, \beta \in \mathbb{C}$ and $\sigma = 0$ then $f_n^{\mathrm{sp}}(t) = f(t)$).

Solution
If $f(t) = \alpha t + \beta$ and $\sigma = 0$ then $Y_l = \alpha t_l + \beta$. Consider the objective in (1) for some $g \in W$:

$$L(g) \quad = \quad \frac{1}{n} \sum_{l=0}^{n-1} |Y_l - g(t_l)|^2 + \kappa \int_0^1 |g''(t)|^2 \, dt$$

Note that $L(f) = 0$ and $L(g) \ge \|g''\|^2_{L^2([0,T])} \ge 0$, thus if $g''(t) \ne 0$ then $L(g) > 0$. The smoothing spline estimator is therefore linear: $f_n^{\mathrm{sp}}(t) = \operatorname*{argmin}_{f\in W} L(f) = \alpha' t + \beta'$ and since $L(f) = 0$ and $L(\alpha' t + \beta') > 0$ iff $\alpha \ne \alpha'$ or $\beta \ne \beta'$, the smoothing spline estimator recovers f, i.e. $\alpha' = \alpha$ and $\beta' = \beta$.

(b) Suppose next that W is the set of all the functions $f : [0,1] \to \mathbb{C}$ such that

(i) f' is absolutely continuous and

(ii) the periodicity condition is satisfied: $f(0) = f(1), f'(0) = f'(1)$.

Prove that the minimization problem (1) is equivalent to:

$$\min_{\{b_q\}_{q\in\mathbb{Z}}} \sum_{q\in\mathbb{Z}} \left(-2\mathrm{Re}\left(\hat{\theta}_q b_q^*\right) + |b_q|^2 \left(\kappa|a_q|^2 + 1\right)\left[1 + O(n^{-1})\right]\right), \tag{2}$$

where b_q are the Fourier coefficients of f, $\hat{\theta}_q = \frac{1}{n}\sum_{l=0}^{n-1} Y_l \phi_q^*(t_l)$, and a_q is defined as $a_q = -(2\pi q)^2$.

Bonus: prove that the term $O(n^{-1})$ is uniform in $\{b_q\}_{q\in\mathbb{Z}}$, namely that there exists a constant $C > 0$ that does not depend on $\{b_q\}_{q\in\mathbb{Z}}$ and the modulus of the term is bounded by C/n.

Solution

Since $\{b_q\}_{q\in\mathbb{Z}}$ is the Fourier series of f :

$$f(t) = \sum_{q\in\mathbb{Z}} b_q\phi_q(t) \Rightarrow f(t_l) = f(l/n) = \sum_{q\in\mathbb{Z}} b_q\phi_q(l/n)$$

The Fourier series of f'' is then

$$f''(t) = \sum_{q\in\mathbb{Z}} b_q\phi_q(t) = \sum_{q\in\mathbb{Z}} -(2\pi q)^2 b_q\phi_q(t) = \sum_{q\in\mathbb{Z}} a_q b_q\phi_q(t)$$

By Parseval's identity:

$$\int_0^1 |f''(t)|^2 dt = \sum_{q\in\mathbb{Z}} |a_q|^2|b_q|^2$$

Plugging this into the objective (1) gives

$$L(f) \quad = \quad \frac{1}{n}\sum_{l=0}^{n-1} |Y_l - \sum_{q\in\mathbb{Z}} b_q\phi_q(t_l)|^2 + \kappa\sum_{q\in\mathbb{Z}} |a_q|^2|b_q|^2 \tag{3}$$

Rewrite the first summand in (3) and plug in $\hat{\theta}_q$:

$$\begin{aligned}
\frac{1}{n}\sum_{l=0}^{n-1} |Y_l - \sum_{q\in\mathbb{Z}} b_q\phi_q(t_l)|^2 &= \frac{1}{n}\sum_{l=0}^{n-1}|Y_l|^2 - 2\sum_{q\in\mathbb{Z}} \mathrm{Re}\left(b_q^*\frac{1}{n}\sum_{l=0}^{n-1} Y_l\phi_q^*(t_l)\right) \\
&\quad + \sum_{q_1,q_2\in\mathbb{Z}} b_{q_1} b_{q_2}^* \left(\frac{1}{n}\sum_{l=0}^{n-1}\phi_{q_1}(t_l)\phi_{q_2}^*(t_l)\right) \\
&= \frac{1}{n}\sum_{l=0}^{n-1}|Y_l|^2 - 2\mathrm{Re}\left(\sum_{q\in\mathbb{Z}}\hat{\theta}_q b_q^*\right) + \sum_{q_1,q_2\in\mathbb{Z}} b_{q_1} b_{q_2}^* \mathbf{1}_{n|(q_1-q_2)},
\end{aligned} \tag{4}$$

where for a condition A : $\mathbf{1}_A = \begin{cases} 1 & \text{if } A \text{ holds} \\ 0 & \text{otherwise}\end{cases}$, $n|m$ denotes that n divides m, i.e. $m = nk$ for some $k \in \mathbb{Z}$ and we used the following property of $\phi_q(t_l)$ (that was also derived in the lecture):

$$\begin{aligned}
\frac{1}{n}\sum_{l=0}^{n-1}\phi_{q_1}(t_l)\phi_{q_2}^*(t_l) &= \frac{1}{n}\sum_{l=0}^{n-1} e^{i2\pi(q_1-q_2)l/n} \\
&= \begin{cases} 1, & \text{if } q_1 - q_2 = kn \text{ for some } k \in \mathbb{Z} \\ \frac{1-e^{i2\pi(q_1-q_2)}}{1-e^{i2\pi(q_1-q_2)/n}}, & \text{otherwise}\end{cases} \\
&= \mathbf{1}_{n|(q_1-q_2)}.
\end{aligned}$$

Continuing from (4) and rearranging the summation indices leads to

$$
\begin{aligned}
L(f) - \frac{1}{n}\sum_{l=0}^{n-1}|Y_l|^2 + 2\mathrm{Re}\left(\sum_{q\in\mathbb{Z}}\hat{\theta}_q b_q^*\right) \quad &- \quad \kappa\sum_{q\in\mathbb{Z}}|a_q|^2|b_q|^2 & (5)\\
&= \sum_{q_1,q_2\in\mathbb{Z}} b_{q_1}b_{q_2}^*\mathbf{1}_{n|(q_1-q_2)} \\
&= \sum_{q\in\mathbb{Z}}|b_q|^2 + \sum_{q_1\in\mathbb{Z}}\sum_{k\in\mathbb{Z}\setminus\{0\}} b_{q_1}b_{q_1+kn}^* \\
&= \sum_{q\in\mathbb{Z}}|b_q|^2 + \sum_{\substack{q\in\mathbb{Z}:\\ |q|>n/2}}\sum_{k\in\mathbb{Z}\setminus\{0\}} b_q b_{q+kn}^* + \sum_{\substack{q\in\mathbb{Z}:\\ |q|\leq n/2}}\sum_{k\in\mathbb{Z}\setminus\{0\}} b_q b_{q+kn}^* \\
&\leq \sum_{q\in\mathbb{Z}}|b_q|^2 + \sum_{\substack{q\in\mathbb{Z}:\\ |q|>n/2}}\sum_{q'\in\mathbb{Z}}|b_q||b_{q'}| + \sum_{\substack{q\in\mathbb{Z}:\\ |q|\leq n/2}}\sum_{\substack{q'\in\mathbb{Z}\\ |q'|\geq n/2}}|b_q||b_{q'}| \\
&\leq \sum_{q\in\mathbb{Z}}|b_q|^2 + 2\left(\sum_{\substack{q\in\mathbb{Z}:\\ |q|\geq n/2}}|b_q|\right)\left(\sum_{q\in\mathbb{Z}}|b_q|\right) & (6)
\end{aligned}
$$

Consider the terms that form the second summand:

$$
\begin{aligned}
\sum_{q\in\mathbb{Z}}|b_q| &= |b_0| + \sum_{q\in\mathbb{Z}\setminus\{0\}}(|b_q||a_q|)\frac{1}{|a_q|} \\
&\leq |b_0| + \sqrt{\sum_{q\in\mathbb{Z}}|b_q|^2|a_q|^2}\sqrt{\sum_{q\in\mathbb{Z}\setminus\{0\}}\frac{1}{(2\pi q)^4}} \\
\sum_{\substack{q\in\mathbb{Z}:\\ |q|\geq n/2}}|b_q| &= \sum_{\substack{q\in\mathbb{Z}:\\ |q|\geq n/2}}(|b_q||a_q|)\frac{1}{|a_q|} \\
&\leq \sqrt{\sum_{q\in\mathbb{Z}}|b_q|^2|a_q|^2\sum_{\substack{q\in\mathbb{Z}:\\ |q|\geq n/2}}\frac{1}{(2\pi q)^4}} \\
&\leq \frac{1}{n/2}\sqrt{\sum_{q\in\mathbb{Z}}|b_q|^2|a_q|^2}\sqrt{\sum_{q\in\mathbb{Z}\setminus\{0\}}\frac{1}{(2\pi q)^4}}
\end{aligned}
$$

Taking the product of the above and letting $C^2 = \sum_{q\in\mathbb{Z}\setminus\{0\}} \frac{1}{(2\pi q)^4} = \frac{1}{720}$ gives

$$\begin{aligned}
\left(\sum_{\substack{q\in\mathbb{Z}:\\ |q|\geq n/2}} |b_q|\right)\left(\sum_{q\in\mathbb{Z}} |b_q|\right) &= \frac{1}{n/2}|b_0|\sqrt{\sum_{q\in\mathbb{Z}} |b_q|^2|a_q|^2}C + \frac{1}{n/2}\sum_{q\in\mathbb{Z}} |b_q|^2|a_q|^2 C^2 \\
&\leq \frac{4}{n}\left(|b_0|^2 + C^2\sum_{q\in\mathbb{Z}} |b_q|^2|a_q|^2\right) + \frac{2C^2}{n}\sum_{q\in\mathbb{Z}} |b_q|^2|a_q|^2 \\
&= \frac{2}{n}\sum_{q\in\mathbb{Z}} |b_q|^2(1+6C^2|a_q|^2) \\
&\leq \max\{2, 12C^2/\kappa\}/n\sum_{q\in\mathbb{Z}} |b_q|^2(1+\kappa|a_q|^2)
\end{aligned}$$

Plugging this result into (6) gives that

$$L(f) \quad = \quad \frac{1}{n}\sum_{l=0}^{n-1}|Y_l|^2 - 2\mathrm{Re}\left(\sum_{q\in\mathbb{Z}}\hat{\theta}_q b_q^*\right) + \sum_{q\in\mathbb{Z}}|b_q|^2(1+\kappa|a_q|^2)\left(1+\frac{\max\{4, 24C^2/\kappa\}}{n}\right) \tag{7}$$

Note that since $\frac{1}{n}\sum_{l=0}^{n-1}|Y_l|^2$ does not depend on b_q, minimizing $L(f)$ is equivalent to minimizing $L(f) - \frac{1}{n}\sum_{l=0}^{n-1}|Y_l|^2$. Finally noting that $\frac{\max\{4,24C^2/\kappa\}}{n} = O(1/n)$ gives the desired result.
Bonus:
Note that we $O(1/n)$ term is $\frac{\max\{4,24C^2/\kappa\}}{n} = C'/n$ for some constant C', so the $O(1/n)$ term is uniform in $\{b_q\}_{q\in\mathbb{Z}}$.

(c) Assume now that the term $O(n^{-1})$ in (2) is negligible. Formally replacing it by 0, find the solution of (2) and conclude that the periodic spline estimator is approximately equal to a weighted projection estimator:

$$f_n^{\mathrm{sp}}(x) \approx \sum_{q\in\mathbb{Z}} \lambda_q\hat{\theta}_q\phi_q(t)$$

with weights λ_q written explicitly.
Solution:
If the $O(1/n)$ term is negligible, the problem (2) decomposes over $b_q \in \mathbb{C}$ and the optimal Fourier coefficients coefficients in (2) are:

$$b_q^{\mathrm{opt}} \quad = \quad \underset{b_q\in\mathbb{Z}}{\mathrm{argmin}}\left(-2\mathrm{Re}\left(\hat{\theta}_q b_q^*\right) + |b_q|^2\left(\kappa|a_q|^2+1\right)\right) = \frac{\hat{\theta}_q}{1+\kappa|a_q|^2}$$

The optimal spline estimator is then

$$f_n^{\mathrm{sp}}(t) \approx \sum_{q\in\mathbb{Z}} b_q^{\mathrm{opt}}\phi_q(t) = \sum_{q\in\mathbb{Z}}\frac{1}{1+\kappa|a_q|^2}\hat{\theta}_q\phi_q(t)$$

(The $\approx$ comes from the fact that we neglected the $O(1/n)$ term) $\lambda_q = \frac{1}{1+\kappa|a_q|^2}$

(d) Use (c) to show that for sufficiently small κ the spline estimator f_n^{sp} is approximated by the kernel estimator:

$$f_n(t) = \frac{1}{nh}\sum_{l=0}^{n-1} Y_l K\left(\frac{t_l - t}{h}\right),$$

where $h = \kappa^{1/4}$ and K is the Silverman kernel:

$$K(u) = \int_{-\infty}^{\infty} \frac{\cos(2\pi t u)}{1+(2\pi t)^4} dt.$$

Solution
If h is small enough to justify the approximation

$$K\left(u\right) \approx \sum_{q\in\mathbb{Z}} \frac{\cos(2\pi q h u)}{1+(2\pi q h)^4} h$$

then one can rewrite the kernel estimator as

$$\begin{aligned}
f_n(t) &\approx \frac{1}{n}\sum_{l=0}^{n-1} Y_l K\left(\frac{t_l - t}{h}\right) \\
&= \frac{1}{n}\sum_{l=0}^{n-1} Y_l \sum_{q\in\mathbb{Z}} \frac{\cos\left(2\pi q h \frac{t_l - t}{h}\right)}{1+(2\pi q h)^4} \\
&= \frac{1}{n}\sum_{l=0}^{n-1} Y_l \sum_{q\in\mathbb{Z}} \frac{\cos(2\pi q (t_l - t))}{1+\kappa|a_q|^2} \\
&= \frac{1}{n}\sum_{l=0}^{n-1} Y_l \sum_{q\in\mathbb{Z}} \frac{e^{2\pi q(t-t_l)} + e^{-2\pi q(t-t_l)}}{2(1+\kappa|a_q|^2)} \\
&= \frac{1}{n}\sum_{l=0}^{n-1} Y_l \sum_{q\in\mathbb{Z}} \frac{e^{2\pi q(t-t_l)}}{1+\kappa|a_q|^2} \\
&= \frac{1}{n}\sum_{l=0}^{n-1} Y_l \sum_{q\in\mathbb{Z}} \frac{\phi_q(t)\phi_q^*(t_l)}{1+\kappa|a_q|^2} \\
&= \sum_{q\in\mathbb{Z}} \frac{1}{1+\kappa|a_q|^2} \left(\frac{1}{n}\sum_{l=0}^{n-1} Y_l \phi_q^*(t_l)\right) \phi_q(t) \\
&= \sum_{q\in\mathbb{Z}} \frac{1}{1+\kappa|a_q|^2} \hat{\theta}_q \phi_q(t) \approx f_n^{\mathrm{sp}}(t)
\end{aligned}$$