EE 378A Statistical Signal Processing Professor Andrea Montanari

Homework 3

Due on: 05/09/2022

Problem 1

Let

$$\Theta_{N,q} := \left\{\theta \in \mathbb{R}^N : \|\theta\|_q \le r\right\},$$

where, for $q > 0$, we defined the q-th pseudonorm as

$$\|\theta\|_q := \left(\sum_{i=1}^{N} |\theta_i|^q\right)^{1/q}.$$

We observe $y = \theta + \sigma g$ for $g \sim \mathcal{N}(0, I_N)$ and consider the minimax risk

$$\bar{R}_{nl}(N; r, \sigma) := \inf_{\hat{\theta}:\mathbb{R}^N \to \mathbb{R}^N} \sup_{\theta \in \Theta_{N,q}(r)} \mathbb{E}\left\{\left\|\theta - \hat{\theta}(y)\right\|^2\right\}$$

(1) Use scale invariance to show that

$$\bar{R}_{nl}(N; ar, a\sigma) = a^2 \bar{R}_{nl}(N; r, \sigma).$$

Deduce that

$$\bar{R}_{nl}(N; r, \sigma) = \bar{R}_{nl}\left(N; \frac{r}{\sigma}, 1\right)\sigma^2 =: R_{nl}\left(N; \frac{r}{\sigma}\right)\sigma^2$$

(2) Consider the soft thresholding denoiser at level λ. Show that

$$R_{nl}(N; r) \le \sum_{i:|\theta_i| \ge c_0\lambda} (1 + \lambda^2) + \sum_{i:|\theta_i| < c_0\lambda} c_1 e^{-\lambda^2/2}$$

for suitable constants c_0, c_1.

(3) Optimize over λ to show that for $N/r^q \to \infty,\ q \in (0, 2)$ we have

$$R_{nl}(N; r) \le C r^q \left(2 \log \frac{N}{r^q}\right)^{1-q/2},$$

substitute in the expression for $\bar{R}_{nl}(N; r, \sigma)$.

(4) Consider next the linear risk

$$R_l(N; r, \sigma) := \inf_{M \in \mathbb{R}^{N \times N}} \sup_{\theta \in \Theta_{N,q}(r)} \mathbb{E}\left\{\|My - \theta\|_2^2\right\}.$$

Compute the upper bound $R_l^d(N; r, \sigma)$ on R_l, which is obtained by restricting M to be diagonal.

(5) **Optional** How does R_l^d compare to R_l?