~~M~~ 3/28/2022 EE378A

Statistical Signal Processing

- Information contained in
- functions/signals
- noise/corruptions

* Linear methods (~~R~~ Denoising)
* Nonlinear/sparsity methods ~~(Con~~
  (Compression/Denoising/Inverse Problems)
* Bayesian methods
* Deep Learning

Focus on ~~theorem~~ math/structure

- 3 HW
- 1 ~~Reading~~ Literature + project

Spaces of functions

$f : [0,T] \to \mathbb{C}$ or $\to \mathbb{R}$

Hilbert space $\mathcal{H}$

- Vector space on $\mathbb{C}$ ($\mathbb{R}$)
  (can do $af + bg$ $\forall a, b \in \mathbb{C}$, $f, g \in$
  $\in \mathcal{H}$
- Scalar product ($\langle f, g \rangle$ = linear in first arg.)
  $f, g \mapsto \langle f, g \rangle \in \mathbb{C}$ ($\in \mathbb{R}$)
  $\left.\begin{array}{l} \langle f, g \rangle = \langle g, f \rangle^* \\ \langle f, f \rangle \geq 0 \quad \forall f \end{array}\right] \|f\| =$
  $= 0 \Rightarrow f = 0$
- Complete (every Cauchy sequence converges)
- Will assume separable (countable ~~basi~~ dense s

- Consequences

$\exists$ orthonormal basis $(\varphi_n)_{n \in \mathbb{N}}$

$$\langle \varphi_n, \varphi_m \rangle = 1_{n=m}$$

$$\operatorname{span}\left(\left\{ \sum_{\ell=1}^{N} a_\ell \varphi_\ell \;:\; a_\ell \in \mathbb{C},\ N \in \mathbb{N} \right\}\right) = \mathcal{H}$$

- For any continuous $\underbrace{\text{functional}}_{\text{linear}}$

$$F : \mathcal{H} \to \mathbb{C}$$
$$\varphi \longmapsto F(\varphi)$$

$|F(\varphi)| \leq C\|\varphi\|$ ← Continuity

$\exists$ $v \in \mathcal{H}$ such that

$$F(\varphi) = \langle \varphi, v \rangle.$$

- $$f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n$$

Examples

$$- \; L^2([0,T]) = \left\{ f : [0,T] \to \mathbb{C} \;:\; \int_0^T |f(t)|^2 \, dt < \infty \right.$$

$$\langle f, g \rangle_{L^2} := \int_0^T f(t)\, g^*(t)\, dt$$

Orthonormal basis

$$\varphi_n(t) = \frac{1}{\sqrt{T}} e^{i \frac{2\pi n}{T} t} \qquad n \in \mathbb{Z}.$$

$*$

Polynomials: Legendre

- Sobolev space $\quad k \in \mathbb{N} \quad k \geq 1$

$$W^{k,2}([0,T]) = \left\{ f : [0,T] \to \mathbb{C} \;:\; \int_0^T |f(t)|^2 dt + \int_0^T |f^{(k)}(t)|^2 dt < \infty \right\}$$

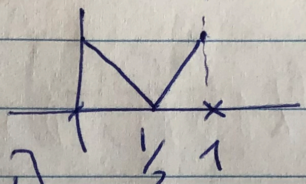
(weak) k-th derivative

~~Periodic~~ Periodic: in part $f^{(\ell)}(0) = f^{(\ell)}(T)$

~~$W^{k,2}$~~ $\forall\, \ell \le k-1$.

(Weak derivative vs derivative

$T=1$, $f(t) = |t - \tfrac{1}{2}|$



$$f'(t) = \begin{cases} 1 & \text{if } t \in [\tfrac{1}{2}, 1] \\ -1 & \text{if } t \in [0, \tfrac{1}{2}) \end{cases}$$

$$\int_0^1 f'(t)\,\varphi(t)\,dt = -\int_0^1 f(t)\,\dot\varphi(t)\,dt$$

$\forall$ smooth periodic φ

$f(t) = |t - \tfrac{1}{4}|$ weakly diff at 0 and 1/4 not at 0, 1.)

$$\langle f, g\rangle = \int_0^T f(t)\,g^*(t)\,dt + \int_0^T f^{(k)}(t)\,g^{(k)}(t)^*\,dt$$

Basis φ_n as in $(*)$. (trigon.)

$\uparrow$ orthogonal, not orthonormal

$\cancel{\langle \varphi_n, \varphi_m \rangle_W}$ $\qquad \varphi_n(t) = \frac{1}{\sqrt{T}} e^{i \frac{2\pi n}{T} t}$

$$\varphi_n^{(k)}(t) = \left(i \frac{2\pi n}{T}\right)^k \underbrace{\frac{1}{\sqrt{T}} e^{i \frac{2\pi n}{T} t}}_{\varphi_n(t)}$$

$$\langle \varphi_n, \varphi_m \rangle_{W^{k,2}} = \left[1 + \left(\frac{2\pi n}{T}\right)^{2k}\right] \delta_{m,n}$$

$$\langle f, g \rangle_{W^{k,2}} = \sum_{n \in \mathbb{Z}} \left(1 + \left(\frac{2\pi n}{T}\right)^{2k}\right) a_n^* b_n$$

f: $\sum_n a_n \varphi_n$ $\qquad g$: $\sum_n b_n \varphi_n$

Sobolev ball

$$W^{k,2}([0,T], R) = \{ f : \|f\|_{W^{k,2}} \le R \}$$

$$\simeq \cancel{\Theta^{(k)}(T,R) := \{ \theta \in \mathbb{R}^{\mathbb{Z}} \sum_n \left(1 + \left(\frac{2\pi n}{T}\right)^{2k}\right) |\theta_n|^2}$$

$\cancel{W^{k,2}([0,T]}$,

$$W^{k,2}([0,T],R) \cong \Theta^{k,2}(T,R)$$

(*) $$\Theta^{k,2}(T,R) := \left\{ \theta \in \mathbb{R}^{\mathbb{Z}} : \sum_{n \in \mathbb{Z}} \left(1 + \left(\frac{2\pi n}{T}\right)^{2k}\right) \cdot |\theta_n|^2 \leq R \right.$$

The ^{k,2} Sobolev ball is an ellipsoid.

note that (*) makes sense for any k (not nec integer). Hence we can define $\Theta^{s,2}(T,R)$ for any $s \in \mathbb{R}$ and define $W^{s,2}([0,T],R)$ as the set of functions whose Fourier series is in $\Theta^{s,2}(T,R)$.

$s \to$ smoothness