(Linear) inverse problems

Unknown signal f

(eg $f \in L^2([0,1])$, $f \in L^2([0,1]\times[0,1])$
...)

Observe for $i = 1, \dots n$

~~$y_i = \langle a_i,$~~ $y_i = \langle f, a_i \rangle_{L^2} + z_i$

z_i independent noise

want to reconstruct f

$$f(x) = \sum_{\ell=1}^{N} \theta_\ell^* \varphi_\ell(x)$$

suppose I can truncate at some level

orthonormal basis

$$y_i = x_i^T \theta_* + z_i$$

$x_i = (\langle \varphi_1, a_i \rangle, \ldots \langle \varphi_N, a_i \rangle) \in \mathbb{R}^N$

$\rightarrow \quad y = X\theta_* + z$

entries $O(1)$

suppose X has orthogonal columns

$$X^T X = n I_N \qquad (n \geq N)$$

wlog can comput

$$\tilde{y} = \frac{1}{n} X^T y \qquad \& \quad z \sim N(0, \sigma^2 I_n)$$

$$\tilde{y} = \theta_* + g \qquad g \sim N(0, \frac{\sigma^2}{n} I_N)$$

At this point do denoising (ST)

$$\hat{\theta} = \eta(\tilde{y}; \lambda)$$

; Assume θ ksparse

<u>We proved</u>

$$R(\sigma^2; N, n) \approx \frac{2\sigma^2 k}{N} \log \frac{N}{k} (1 + o(1))$$

Can we achieve the same for ~~N~~ $\leq n$

$N > n$ (or $N \gg n$?)

in particular non-orthogonal columns?

How?

recall that for ($y \in \mathbb{R}^N$)

$\hat{\theta}(y)$ ~~$\eta(y; \lambda) = \arg\min \{ \frac{1}{2} \|y - x)$~~

ok

$$\hat{\theta} = \eta(\tilde{y}; \tilde{\lambda}) = \arg\min_{\theta} \left\{ \frac{1}{2} \|\tilde{y} - \theta\|_2^2 + \tilde{\lambda} \|\theta\|_1 \right\}$$

$$= \arg\min_{\theta} \left\{ \frac{1}{2} \left\| \frac{1}{n} X^T y - \theta \right\|_2^2 + \tilde{\lambda} \|\theta\|_1 \right\}$$

$$\frac{1}{n} \|Xv\|_2^2 = \frac{1}{n} \langle v, X^T X v \rangle = \|v\|^2$$

$$\hat{\theta}(y) = \arg\min_{\theta} \left\{ \frac{1}{2n} \left\| X\left(\frac{1}{n} X^T y - \cancel{\theta}\right) \right\|^2 + \lambda \|\theta\|_1 \right\}$$

$$= \arg\min \left\{ \frac{1}{2n} \| P_X y - X\theta \|^2 + \lambda \|\theta\|_1 \right\}$$

$$= \arg\min \left\{ \frac{1}{2n} \| y - X\theta \|^2 + \lambda \|\theta\|_1 \right\}$$

LASSO / Basis pursuit. Makes perfect sense even if ~~X~~ $n < N$!

Does it achieve the "ideal" risk.

~~Def X satisfies the Restri~~

Def For $S \subseteq [N]$, $\alpha \in$, define the cone

$$\mathcal{C}(S,\alpha) := \{ v \in \mathbb{R} : \|v_{S^c}\|_1 \leq \alpha \|v_S\|_1 \}$$

~~We~~

Def We say that $X \in \mathbb{R}^{n \times N}$ satisfies Restricted Eigenvalue with parameters $S \subseteq [N]$, $\alpha, \gamma > 0$ if

$$v \in C(S, \alpha) \Rightarrow \frac{1}{n}\|Xv\|^2 \geq \gamma \|v\|^2.$$

Theorem Assume ~~$X \in RE$~~ ~~(for suitable~~ con ~~θ_*~~

$\mathrm{supp}(\theta_*) = S$ $|S| \leq k$. and X satisfies RE with parameters $\|Xe_i\|^2 \leq n$

$\alpha \leq \alpha_0 = 5$, $\gamma \geq \gamma_0 = \frac{9}{10}$. Then there exists ~~an~~ absolute constants C_0, C_1 such that ~~x_*~~ for

$$\lambda \geq \sqrt{\frac{C_0 \log N}{n}} = \lambda_*$$

We have ~~$\|\hat{\theta}(\lambda) - \theta_*\|^2$~~

$$\|\hat{\theta}(\lambda) - \theta_*\|_2^2 \leq C_1 \oint k \lambda^2$$

with probability at least $1 - C_2 n^{-3}$. $\square$

<u>Thm</u> If ~~(X_{ij})~~ X has iid subgaussian rows $\mathbb{E} X_i X_i^T = I_N$, $\mathbb{E} X_i = 0$

$\mathbb{E} \exp\{\langle u X_i \rangle\} \leq \exp\{c \frac{\|u\|^2}{2}\}$

then whp it satisfies ~~RE~~ restricted eigenvalue condition with parameters $\alpha_0 \gamma_0$, provided

$$\cancel{k \leq \tilde{c} k}$$

$$n \geq \tilde{c} k \log \frac{N}{k}$$

[Candés, Tao

Bickel, Ritov, Tsybakov

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Proof $\quad \hat{\theta} = \underset{\theta}{\operatorname{argmin}} \left\{ \frac{1}{2n} \|y - X\theta\|_1^2 + \lambda \|\theta\|_1 \right\}$

$$\frac{1}{2n} \|z \cancel{\theta} - X(\hat{\theta} - \theta_*)\|^2 + \lambda \|\hat{\theta}\|_1 \leq \frac{1}{2n} \|z\|_2^2 + \|\theta_*\|_1$$

~~subst~~ $\hat{\theta} := \theta_* + u$

$$-\frac{1}{n} \langle z, Xu \rangle + \frac{1}{2n} \|Xu\|^2 + \lambda \left(\|\theta_* + u\|_1 - \|\theta_*\|_1 \right) \leq 0$$

& let ~~sign(θ_*) =~~ { ~~$s(\theta_*$~~

$$s(\theta_*) = \begin{cases} +1 & \text{if } \theta_{*,i} > 0 \\ -1 & \quad\quad\ < 0 \\ 0 & \quad\quad \text{ow} \end{cases}$$

$$\frac{1}{2n} \|Xu\|^2 + \lambda \left(\|\theta_* + u\|_1 - \|\theta_*\|_1 - \langle s(\theta_*), u \rangle \right)$$

$$\leq \frac{1}{n} \langle X^T z, u \rangle - \lambda \langle s(\theta_*), u \rangle$$

$S = supp(\theta_*)$

$\leq \|\frac{1}{n}X^T z\|_\infty \|u\|_1 + \lambda\|u_S\|_1 =: \cancel{10\lambda}$

$s(\theta_*) \in \partial\|\theta\|_1 \Big|_{\theta=\theta_*}$ ← subdifferential

$\|\theta_* + u\|_1 - \|\theta_*\|_1 - \langle s(\theta_*), u\rangle \geq \|u_{S^c}\|_1 \geq 0$

Let $\lambda(z) := 4\|\frac{1}{n}X^T z\|_\infty$

$\frac{1}{2n}\|X^T u\|_2^2 \leq \frac{\lambda(z)}{4}\|u\|_1 + \lambda\|u_S\|_1$

$\|u_{S^c}\|_1 \leq \frac{\lambda(z)}{4\lambda}\|u\|_1 + \|u_S\|_1$

On the event $\boxed{\lambda(z) \leq \lambda}$

$\frac{1}{2n}\|X^T u\|_2^2 \leq \lambda\left(\frac{5}{4}\|u_S\|_1 + \|u_{S^c}\|_1\right)$

$\frac{3}{4}\|u_{S^c}\|_1 \leq \frac{5}{4}\|u_S\|_1$

$$\begin{cases} \|u_{S^c}\|_1 \le \frac{5}{3}\|u_S\|_1 \\ \frac{1}{2n}\|X^T u\|_2^2 \le \lambda \cdot \frac{5\cdot 7}{12} \|u_S\|_1 \\ \le 3\lambda \|u_S\|_1 \end{cases}$$

By the RE condition

$$\cancel{\frac{9}{20}\|u_S\|_2^2} \; \frac{9}{20}\|u\|_2^2 \le 3\lambda\|u_S\|_1 \le 3\lambda\sqrt{k}\,\|u_S\|_2$$

$$\frac{9}{20}\|u_S\|_2^2 \le$$

$$\|u_S\|_2 \le \frac{20}{9}\cdot 3\lambda\sqrt{k} \le 6\lambda\sqrt{k}$$

$$\|u\|_2^2 \le (3\lambda\sqrt{k})^2 \cdot \left(\frac{20}{9}\right)^2 \le Ck\lambda^2$$

$$\|\hat{\theta}_\lambda - \theta\|_2^2 \le Ck\lambda^2$$

Remains to decide whether the event $\{\lambda(z) \leq \lambda_*\}$ holds whp.

$$\mathbb{P}\left(\frac{1}{n}\|X^T z\|_\infty \geq t\right) =$$

$$= \mathbb{P}\left(\max_{i \leq N} \frac{1}{n}\langle \tilde{x}_i, z\rangle \geq t\right)$$

$$\leq N \max_i \mathbb{P}\left(\frac{\langle \tilde{x}_i, z\rangle}{\sqrt{n}} \leq t\sqrt{n}\right)$$

$$\leq N \exp\left(-\frac{cnt^2}{2\sigma^2}\right)$$ z subgaussian

$$t = \lambda_* = c_0\sqrt{\frac{\sigma^2 \log N}{n}}$$

$$\leq N \exp\{-cc_0 \log N\} \leq N^{-10}.$$

For c_0 ~~big~~ enough.