3/30/2022

We are interested in Hilbert spaces of functions such that the evaluation map $t \mapsto f(t)$ is continuous.

Before diving into this, let us make sure we understand why this is important.

Assume we do ridge regr in $L^2([0,T])$

$y_i = f(t_i) + z_i$

$\frac{1}{n}\sum_{i=1}^{n}(y_i - f(t_i))^2$

$$\hat{f} := \underset{f}{\operatorname{argmin}} \left\{ \|y - f(t)\|_{n,2}^2 + \lambda \|f\|_{L^2}^2 \right\}$$

The minimizer is $f(t) = \begin{cases} y_i & \text{if } t = t_i \\ 0 & \text{ow.} \end{cases}$

(Strictly speaking $f(t)$ does not even make sense in L^2)

Reproducing Kernel Hilbert Spaces ~~are~~

(RKHS)

- these are defined as subspaces of $L^2([0,T])$ for which $t \mapsto f(t)$ is cont.
- Interesting independently, so let us ~~get~~ ~~into~~ take a detour.
- Applications to ML, ... (t does not need to be scal.)

$\mathcal{H}$ RKHS

$\Rightarrow \exists\, h_t \in \mathcal{H} \quad \forall t \in [0,T]$ st

$\forall f \in \mathcal{H}$ & $f(t) = \langle f, h_t \rangle_{\mathcal{H}}$

$\Rightarrow$ Can define a kernel

$$k(t,s) := \langle h_t, h_s \rangle_{\mathcal{H}}.$$

~~Properties~~ $\cancel{= \|h_t\|^2_{\mathcal{H}}}$

$\cancel{- \; 0 \le K(t,t) < \infty}$

$\cancel{\Rightarrow \quad |K(t,s)| \le \sqrt{K(t,t)K(s,s)}}$

Properties $\quad - \; K(t,s) = K(s,t)^*$

$- \quad 0 \le K(t,t) = \|h_t\|^2_{\mathcal{H}} < \infty.$

Will assume for simplicity $K(t,t) \le C < \infty$

$\Rightarrow |K(t,s)| \le C \quad \forall t,s.$

$- \quad \cancel{h_t \sum_{j=1}^{m} K(t_i,t_j)}$

$K \succeq 0 \quad [\forall t_1 \dots t_m \quad [K(t_i,t_j)]_{i,j \le m} \succeq 0]$

$- \quad h_{t_0}(t) = K(t_0,t)$

$= \langle h_{t_0}, h_t \rangle$

$$- f(t) = \sum_{i=1}^{m} c_i h_{t_i}(t) = \sum_{i=1}^{m} c_i K(t_j, t_i) \in \mathcal{H}.$$

- ~~If (f,g~~

Since K symmetric, trace class

$\exists\,(\varphi_n)$ orthonormal basis of $L^2([0,T])$ st

$$K(t,s) = \sum_{n=1}^{\infty} \lambda_n^2 \varphi_n(t)^* \varphi_n(s)^*$$

[corresponding integral operator $\mathbb{K}$]

$$\mathrm{Tr}(\mathbb{K}) = \int_0^T K(t,t)\,dt = \sum_{n=1}^{\infty} \lambda_n^2 < \infty.$$

generally assume $\lambda_1^2 \geq \lambda_2^2 \geq \ldots$

Rmk finite linear combinations are dense

$$f(t) = \sum_{n=1}^{m} c_n \underbrace{K(t_i, t)}_{h_{t_i}(t)}$$

suppose this wasn't the case.

Then there would exist $g \in \mathcal{H}$ st $g \neq 0$

$\langle g, h_{t_0} \rangle = 0 \quad \forall t_0$

$\Rightarrow \quad g(t_0) = 0 \Rightarrow \quad g = 0$

Claim $\quad f = \sum_n a_n \varphi_n, \quad g = \sum_n b_n \varphi_n$

$$\Rightarrow \quad \langle f, g \rangle_{\mathcal{H}} = \sum_{n=1}^{\infty} \frac{a_n b_n^*}{\lambda_n^2}$$

(RHS)

Indeed this is a good scalar product.
Sufficient to check identitly on
a denseset.

$\Rightarrow$ sufficient to check for $f = h_{t_1}, g = h_{t_2}$

$\langle h_{t_1}, h_{t_2} \rangle_{\mathcal{H}} = K(t_1, t_2)$ by def

$h_{t_1}(t) = K\ell \; \langle h_{t_1}, h_t \rangle = k(t_1, t)$

$$= \sum_{n=1}^{\infty} \underbrace{\lambda_n^2 \varphi_n^*(t_1)}_{a_n} \varphi_n(t)^*$$

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} a_n b_n^* = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \lambda_n^2 \varphi_n^*(t_1) \lambda_n^2 \varphi_n(t_2)$$

$$= \sum_{n=1}^{\infty} \lambda_n^2 \varphi_n(t_1)^* \varphi_n(t_2) = k(t_1 t_2) \, !$$

- So, balls in an RKHS are ellipsoids in $\ell^2(\mathbb{N})$. $\cancel{B_K(\mathbb{R})} := B_K([0,T], R) =$

$$\{f : \|f\|_K^2 \le R^2\} \cong \{\theta \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} \frac{|\theta_n|^2}{\lambda_n^2} \le R^2\}$$

RKHS norm for kernel K

- RKHS norm ~~is determined~~ determined uniquely by the kernel (eigenvalues and eigenvectors).

Relation between RKHS and Sobolev (Periodic functions on $[0,T]$)

Consider

$$K(t,s) = K_0(t-s)$$

~~We with~~ $K_0(t+T) = K_0(t)$

$K(s,t) = K(t,s)^*$

$\Rightarrow \; K_0(-t) = K_0(t)^*$

By periodicity, we can write

$$K_0(t) = \frac{1}{T} \sum_{n \in \mathbb{Z}} \lambda_n^2 \, e^{-i\frac{2\pi n}{T}t}$$

$$\Rightarrow K(t,s) = \sum_{n \in \mathbb{Z}} \lambda_n^2 \, \varphi_n^*(t) \, \varphi_n(s)$$

$$\lambda_n^2 = \int_0^T K_0(t) \, e^{i\frac{2\pi n}{T}t} \, dt$$

$$\Rightarrow B_K([0,T],\mathbb{R}) = W^{\ell,2}([0,T],\mathbb{R})$$

for $$K_0(t) = \frac{1}{T} \sum_{n \in \mathbb{Z}} \left(1 + \left(\frac{2\pi n}{T}\right)^{2\ell}\right)^{-1} e^{-\frac{2\pi i n}{T}t}$$

More generally, we will see that the RKHS ball (behavior of estimation in the) depends on the asymptotics of λ_n

Hence

$\lambda_n \simeq n^{-s}$ $\mathcal{H}$ behaves as the Sobolev space with smoothness s.