Lecture 3

$f : [0,T] \to \mathbb{R}$ periodic

$f \in \mathcal{H}$ RKHS with kernel

$\lambda_n \simeq n^{-s}$ $\quad K(t,s) = K_0(t-s)$ periodic

$y_i = f^*(t_i) + \sigma z_i \qquad z_i \sim N(0,1)$

$0 \leq i \leq n-1 \qquad t_i = \frac{i}{n} T$

~~Equivalent (Le Cam sense)~~

Hereafter wlog $T=1$

$\varphi_q(t) = e^{2\pi i q t} \qquad q \in \mathbb{Z}$

Lemma: define $\varphi_q^{(n)} = \frac{1}{\sqrt{n}}\left(\varphi_q(t_i)\right)_{i\in[0,n-1]} \in \mathbb{C}^n$

Then $\left(\varphi_q^{(n)}\right)_{q\in\{0,\dots n-1\}}$ are an orthonormal basis of $\mathbb{C}^n$.

Proof

$$\langle \varphi_p^{(n)}, \varphi_q^{(n)} \rangle = \frac{1}{n}\sum_{\ell=0}^{n-1} e^{2\pi i (p-q)\frac{\ell}{n}} =$$

$p \neq q$

$$= \frac{1}{n}\,\frac{1-e^{2\pi i(p-q)}}{1-e^{2\pi i(p-q)/n}} = 0$$

$p = q$

$$= 1$$

$\boxtimes$

Wlog we can change basis

$$\tilde{Y}_q = \frac{1}{\sqrt{n}}\langle \varphi_q^{(n)}, Y \rangle$$

$$\tilde{f}_q = \frac{1}{\sqrt{n}}\langle \varphi_q^{(n)}, \overbrace{(f^*(t_\ell))_{0\le\ell\le n-1}}^{f^{(n)}} \rangle$$

for n even

$\tilde{\hat{y}}_q = \tilde{\hat{f}}^*_q + \frac{\sigma}{\sqrt{n}} g_q$ ← Note we can take $q \in \{-\frac{n}{2}+1, \dots \frac{n}{2}\} = S_n$

$(g_q)_{q \in S_n} \sim_{iid} N_{\mathbb{C}}(0,1)$

by invariance of Gaussian distr.

$\sqrt{n}\, \tilde{\hat{f}}^*_q = \langle \varphi^{(n)}_q, f^{(*)} \rangle = \cancel{\phi}$ $\qquad f(t) = \sum_{p \in \mathbb{Z}} \theta_p \, \varphi_p(t)$

$\cancel{\mathbb{C}} = \sum_{p \in \mathbb{Z}} \theta_p \langle \varphi^{(n)}_q, \varphi^{(n)}_p \rangle \frac{1}{\sqrt{n}}$

$\cancel{\hat{f}_q = \sqrt{n}\,\theta_q \text{ for } q}$

$\tilde{\hat{f}}^*_q = \sqrt{n}\,\theta_q + \underbrace{\frac{1}{\sqrt{n}} \sum_{p \in \mathbb{Z} \setminus S_n} \theta_p \langle \varphi^{(n)}_q \varphi^{(n)}_p \rangle}_{\alpha_q}$

$|\alpha_q| \leq \sum_{|p| \geq \frac{n}{2}-1} |\theta_p| \leq \cancel{\sum_{|p| \geq \frac{n}{2}-1} \frac{|\theta_p|}{\alpha_p}}$ ϵ_n

$\leq$

~~Suppose now~~

Projection estimator

$$\hat{f}(t) = \sum_{q \in S_m} \tilde{y}_q \varphi_q(t) \qquad m \le n \quad m \ll n$$

~~$\mathbb{E}\{\hat{f}\}$~~

$$\|\hat{f} - f^*\|^2_{L^2[0,1]} = \sum_{q \in S_m} |\tilde{y}_q - \theta_q|^2 + \sum_{q \notin S_m} |\theta_q|^2$$

$$= \sum_{q \in S_m} \left| \alpha_q + \frac{\sigma}{\sqrt{n}} g_q \right|^2 + \sum_{q \notin S_m} |\theta_q|^2$$

$$\mathbb{E}\{\|\hat{f} - f^*\|^2_{L^2}\} = \frac{m\sigma^2}{n} + \sum_{q \in S_m} |\alpha_q|^2 + \sum_{q \notin S_m} |\theta_q|^2$$

$$\le \frac{m\sigma^2}{n} + \underbrace{\sum_{q \notin S_m} |\theta_q|^2}_{(*)} + \underbrace{m \left[\sum_{q \notin S_n} |\theta_q| \right]^2}_{(**)}$$

Now we bound (*) and (**)

$$(*) \leq \frac{X}{\lambda^2_{m/2}} \lambda^2_{m/2} \sum_{q \notin S_m} \frac{|\theta_q|^2}{\lambda_q^2}$$

$$\leq \|f^*\|_K^2 \cdot \lambda^2_{m/2} \leq C m^{-2s} \|f^*\|_K^2$$

$$(**) \leq m \left(\sum_{q \notin S_n} \frac{|\theta_q|}{\lambda_q} \lambda_q \right)^2 \leq m \left(\sum_{q \notin S_n} \frac{|\theta_q|^2}{\lambda_q^2} \right) \sum_{q \notin S_n} \lambda_q^2$$

$$\leq C m \|f\|_K^2 n^{-2s+1}$$

$$\mathbb{E}\left\{\|\hat{f}-f^*\|_{L_2}^2\right\} \leq \frac{m\sigma^2}{n} + C\|f^*\|_K^2 \left(m^{-2s} + m n^{-2s+1}\right)$$

Balancing we obtain

$$m = \left(\frac{\|f\|_K^2}{\sigma^2}\right)^{1/2s+1} n^{1/2s+1}$$

For $s \geq 2$ third term comparable or negligible hence

(*) $s \geq 1$

$$\mathbb{E}\left(\|\hat{f}-f_*\|_{L^2}^2\right) \leq C\left(\frac{\sigma^2}{n}\right)^{\frac{2s}{2s+1}} \cdot \|f\|_K^{2/2s+1}$$

$$\simeq n^{-2s/(2s+1)}$$

Rmk: Note that for simplicity of notation we wrote this proof as if the ordering of λ_q matched the one of frequencies, but this is not important [can be general.]

Rmk If $|\lambda_m^*| = o(m^{-1})$, the above proof implies

$$\mathbb{E}\left(\|\hat{f}-f^*\|_{L^2}^2\right) \lesssim \frac{\sigma^2 m}{n} + C\|f\|_K^2 \lambda_m^2$$