However if we ~~kne~~ know f has only one discontinuity in $[0,T]$, ~~we s~~ an is C^2 otherwise, we can use $O(1)$ bits to encode the discont. and achieve the C^2 rate otherwise.

[Wavelet bases achieve this type of goal <u>without</u> knowing a priory how many ~~dist~~ discontinuities etc.

Similar problems for denoising, estimation inverse problems etc]

MRA is a nice way to define / construct wavelets

Idea " localize in time and frequ

$\sim$ V_ℓ space of fcts "at scale 2^ℓ"

<u>Def</u> $\{V_\ell\}_{\ell\in\mathbb{Z}}$ closed subspaces of $L^2(\mathbb{R})$ is a multi-resolution approx if the following hold:

(1) $f \in V_\ell \iff f(\cdot - k2^\ell) \in V_\ell \quad \forall k \in \mathbb{Z}$.

(2) $V_{\ell+1} \subseteq V_\ell \quad \forall \ell \in \mathbb{Z}$

(3) $f(\cdot) \in V_\ell \iff f(\cdot/2) \in V_{\ell+1} \quad \forall \ell \in \mathbb{Z}$

(4) $\lim\limits_{\ell\to+\infty} V_\ell = \bigcap\limits_{\ell\in\mathbb{Z}} V_\ell = \{0\}$

(5) $\lim\limits_{\ell\to-\infty} V_\ell = \mathrm{cl}\left(\bigcup\limits_{\ell\in\mathbb{Z}} V_\ell\right) = L^2(\mathbb{R})$

(6) There exists ~~φ~~ $\theta \in L^2(\mathbb{R})$ st $\{\theta(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz basis of V_0 -

linearly indep

Def $\{\varphi_n\}_{n\in\Gamma}$ Γ countable is a Riesz basis of a Hilbert space $\mathcal{H}$ if

$\overline{\mathrm{span}(\{\varphi_n\})} = \mathcal{H}$ and $\forall f \in \mathcal{H}$

$$f = \sum_n a_n \varphi_n$$

$$\Big\|\sum_n a_n$$

$$B^{-1}\sum_n |a_n|^2 \le \|f\| \le A^{-1}\sum_n |a_n|^2$$

Rmk① Typically $\|\varphi_n\| = 1 \Rightarrow A \le 1 \le B$

Rmk② Equivalently define

$$G_N = (\langle \varphi_i, \varphi_j\rangle)_{i,j\le n}$$

$$B^{-1} \le \sigma_{min}(G_N) \le \sigma_{max}(G_N) \le A^{-1}$$

$\forall N$

Interpretation

P_{V_ℓ} orthogonal projector onto V_ℓ

$P_{V_\ell} f$: best approx of V_ℓ at scale 2^ℓ V_ℓ

$\lim_{\ell \to -\infty} \|f - P_{V_\ell} f\|_{L^2} \Rightarrow 0$ gives increasingly accurate approx.

<u>Lemma</u> Let $V_0 := \overline{\text{span}(\theta(\cdot - n))}$

Then $\{\theta(\cdot - n)\}_{n \in \mathbb{Z}}$ is a Riesz basis iff $\forall \omega \in [-\pi, \pi]$

$$\frac{1}{B} \leq \sum_{k=-\infty}^{\infty} |\hat{\theta}(\omega - 2\pi k)|^2 \leq \frac{1}{A}$$

<u>Proof</u> $f \in V_0$

$$f(t) = \sum_{n \in \mathbb{Z}} a_n \theta(t-n)$$

$$f(t) = \int e^{i\omega t} \hat{f}(\omega) \frac{d\omega}{2\pi}$$

$$\hat{f}(\omega) = \int_{\mathbb{R}} e^{-i\omega t} f(t)\, dt$$

$$\hat{f}(\omega) = \int e^{-i\omega t} \sum_n a_n \theta(t-n)\, dt =$$

$$\hat{f}(\omega) = \hat{a}(\omega)\hat{\theta}(\omega)$$

$$\hat{a}(\omega) := \sum_n e^{-i\omega n} a_n \qquad a_n = \int_{[-\pi,\pi]} e^{i\omega n} \hat{a}(\omega) \frac{d\omega}{2\pi}$$

$$\|f\|^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{a}(\omega)|^2 |\hat{\theta}(\omega)|^2\, d\omega =$$

$$= \frac{1}{2\pi} \int_{[-\pi\pi]} |\hat{a}(\omega)|^2 \sum_{k\in\mathbb{Z}} |\hat{\theta}(\omega+2\pi k)|^2\, d\omega$$

If $\sum_{k\in\mathbb{Z}} |\hat{\theta}(\omega+2\pi k)|^2 \in [\cancel{\frac{1}{B}}, \cancel{\frac{1}{BA}}]$

$$\int |\hat{a}(\omega)|^2 \frac{d\omega}{2\pi} \le \|f\|^2 \le \frac{1}{\cancel{A}} \int |\hat{a}(\omega)|^2 \frac{d\omega}{2\pi}$$

Examples of MRA

Ex #1 Piecewise constant function

$$V_0 = \left\{ f(t) = \sum_{k \in \mathbb{Z}} c_k \, 1(t \in (k, k+1]) \; : \; \sum_{k \in \mathbb{Z}} |c_k|^2 < \infty \right\}$$

Define V_ℓ by "scaling"

$$P_{V_\ell} f(t) = \sum_{k \in \mathbb{Z}} c_k^\ell \, 1(t \in (k2^\ell, (k+1)2^\ell])$$

$$c_k^\ell := \int_{k2^\ell}^{(k+1)2^\ell} f(t)\,dt$$

$$\theta(t) = 1(t \in (0,1])$$

orthonormal.

The 6 properties ~~are~~ intuitively ...

Continuous

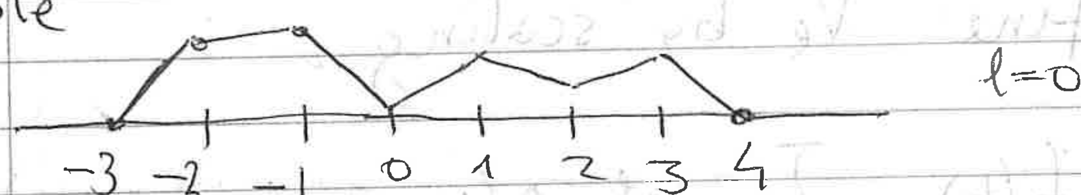
Ex #2 Piecewise linear functions

$$V_\ell = \{ f(x) = \sum_{k\in\mathbb{Z}} h_{\ell,k}(x)\, \mathbb{1}_{x\in(k2^\ell,(k+1)2^\ell]}$$

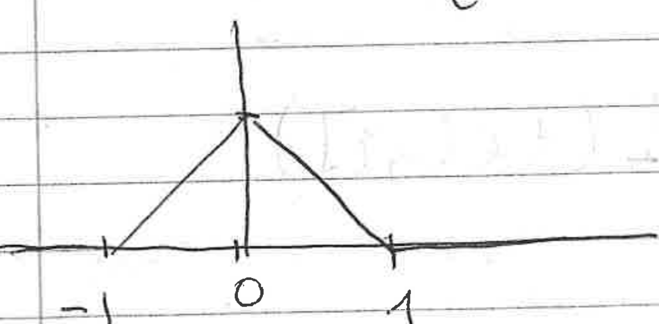
such that $\quad h_{\ell,k}(x) = h^0_{\ell,k} + h^1_{\ell,k}\cdot x$

$$h_{\ell,k+1}((k+1)2^\ell) = h_{\ell,k}((k+1)2^\ell) \}$$

example



$$\theta(t) = \begin{cases} 1-|t| & \text{for } |t|\le 1 \\ 0 & \text{ow} \end{cases}$$



Let us check that span $(\theta(\cdot - k)) = V_0$

redefined

$$f(x) = \sum_{k \in \mathbb{Z}} (h_k^0 + h_k^1 (x-k)) \mathbb{1}_{x \in (k, k+1]}$$

$$= \sum_{m \in \mathbb{Z}} c_m \theta(x-m)$$

c k

$h_k^0 = h_{k-1}^0 + h_{k-1}^1$

continuity.

~~$c_k(1-t)$~~ +

$$c_k(1-t) + c_{k+1} \cdot t = h_k^0 + h_k^1 t$$

$c_{k+1} = c_k + h_k^1$ defines c_{k+1}

$c_k = h_k^0$ satisfied recursively.

Example #3 (generalizes previous 2)

φ_i $V_0 = \{$ splines of degree m with knots at $\mathbb{Z}$ $\}$

$f \in V_0$ (1) must be $(m-1)$-times diff

and

(2) $f\big|_{[k,k+1)}$ is a degree m polyn.

$\varphi(x) = 1_{[0,1)}^{m+1}(x)$ convolution. $(m+1)$ times (box spline)

Example #4 Shannon approx

φ_ℓ $V_\ell = \{ f \in L^2(\mathbb{R}) : \text{supp}(\hat{f}) \subseteq [-2^{-\ell}\pi, 2^{-\ell}\pi] \}$

Let us check:

$f \in V_\ell \iff f(\cdot - k2^{\ell}) \in V_\ell$

~~$g(x) = f(x)$~~

~~Check~~

define $g(x) = f(x - k2^{e})$

$$\hat{g}(\omega) = \int_{\mathbb{R}} f(t - k2^{e}) e^{-i\omega t} dt =$$

$$= \int_{\mathbb{R}} f(t) e^{-i\omega t - i\omega k 2^{e}} dt =$$

$$= \hat{f}(\omega) e^{-i\omega k 2^{e}}$$ ok.

<u>Let us check</u> $f \in V_e \iff f(\cdot/2) \in V_{e+1}$

$g(x) := f(x/2)$

$$\hat{g}(\omega) = \int_{\mathbb{R}} f(t/2) e^{-i\omega t} dt = 2\int f(t) e^{-2i\omega t} dt$$

$$= 2 \hat{f}(2\omega)$$

$\mathrm{supp}(\hat{f}) \subseteq [-2^{-e}\pi, 2^{-e}\pi]$

$\iff \mathrm{supp}(\hat{g}) \subseteq \left[-\frac{2^{-e}\pi}{2}, \frac{2^{-e}\pi}{2}\right]$

$$\hat{\vartheta}(\omega) = 1_{\omega \in [-\pi,\pi]}$$

$$\hat{\vartheta}_n \ \cancel{\text{Four}} := F(\vartheta(\cdot - n))$$

$$\hat{\vartheta}_n(\omega) = \int e^{-i\omega t} \vartheta(t - n)\, dt = e^{-i\omega n} \hat{\vartheta}(\omega)$$

$$= e^{-i\omega n} 1_{\omega \in [-\pi,\pi]}$$

$$\overline{\text{span}\{\hat{\vartheta}_n\}} = L^2([-\pi,\pi])$$

$$\vartheta(t) = \int_{\mathbb{R}} 1_{\omega \in [-\pi\pi]} e^{i\omega t} \frac{d\omega}{2\pi}$$

$$= \int_{-\pi}^{\pi} e^{i\omega t} \frac{d\omega}{2\pi} = \cancel{e^{i\omega\pi}} - e^{-i\omega}$$

$$= \frac{e^{i\pi t} - e^{-i\pi t}}{2\pi \cdot i t} = \frac{\sin(\pi t)}{\pi t}.$$

Construct an orthonormal basis for V_0

<u>Lemma</u> Given a MRA with Riesz basis θ.

Let $\phi = \mathcal{F}^{-1}(\hat{\phi})$

$$\hat{\phi}(\omega) := \frac{\hat{\theta}(\omega)}{\left[\sum_{k\in\mathbb{Z}} |\hat{\theta}(\omega+2\pi k)|^2\right]^{1/2}}$$

Then $\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of V_0.

~~Further~~

<u>Proof</u> (1) Well defined by prev lemma.

$$\hat{\phi}_n(\omega) = e^{-i\omega n}\hat{\phi}(\omega)$$

$$\langle \phi_n, \phi_m \rangle = \frac{\langle \hat{\phi}_n, \hat{\phi}_m \rangle}{2\pi} = \int \frac{|\hat{\theta}(\omega)|^2}{\sum_{k\in\mathbb{Z}} |\hat{\theta}(\omega+2\pi k)|^2} e^{-i\omega(n-m)} \frac{d\omega}{2\pi}$$

$$= \int_{-\pi}^{\pi} \frac{\sum_{\ell\in\mathbb{Z}} |\hat{\theta}(\omega+2\pi\ell)|^2 e^{-i\omega(n-m) - \cancel{i2\pi\ell(n-m)}}}{\sum |\hat{\theta}(\omega+2\pi k)|^2} \frac{d\omega}{2\pi}$$

$$= \int_{-\pi}^{\pi} e^{-i\omega(n-m)} \frac{d\omega}{2\pi} = \delta_{n,m}.$$

$\square$

$$\phi_{e,k}(t) = \frac{1}{\sqrt{2^e}} \phi\left(\frac{t-2^e k}{2^e}\right)$$

For any e $\{\phi_{e,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_e. $\square$

~~Ex #1~~

ϕ is called the scaling function.

Ex #1 piecewise constant fcts

Orthonormal to start with

$$\phi(t) = 1_{t\in[0,1)}$$

$$\hat{\phi}(\omega) = \frac{1-e^{-i\omega}}{i\omega}$$

Ex #2 Piecewise linear

$$\hat{\theta}(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^2 \Rightarrow \ldots$$

Ex #4 Shannon : orthonormal to start with

Conjugate mirror filter

Recall that $V_1 \subseteq V_0$

In particular $\frac{1}{\sqrt{2}} \phi(\cdot/2) \in V_0$

$$\frac{1}{\sqrt{2}} \phi(t/2) = \sum_{k \in \mathbb{Z}} h_k \phi(t-k)$$

Taking Fourier transform

$$\frac{1}{\sqrt{2}} \int \phi(t/2) e^{-i\omega t} dt = \sqrt{2}\, \hat{\phi}(2\omega)$$

$$= \underbrace{\sum_{k \in \mathbb{Z}} h_k e^{-i\omega k}}_{\hat{h}(\omega)} \hat{\phi}(\omega)$$

$$\boxed{\hat{\phi}(2\omega) = \frac{1}{\sqrt{2}} \hat{h}(\omega) \hat{\phi}(\omega)}$$

$$h_k = \frac{1}{\sqrt{2}} \langle \phi(\cdot/2), \phi(\cdot - k) \rangle_{L^2}.$$

Thm The following holds

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega+\pi)|^2 = 2 \qquad (1)$$

$$\hat{h}(0) = \sqrt{2}. \qquad (2)$$

Conversely, assume (1) + (2) and

(3) the $\hat{h}$ is 2π-periodic

(4) $\hat{h}$ cont. diff in a neigh of 0.

(5) $\inf_{\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]} |\hat{h}(\omega)| > 0$

Then

$$\hat{\phi}(\omega) := \prod_{\ell=1}^{\infty} \frac{\hat{h}(2^{-\ell}\omega)}{\sqrt{2}}$$

is the fourier tr. of a scaling function

Wavelets

Note that $V_\ell \subseteq V_{\ell-1}$

So we can decompose

$$V_{\ell-1} = V_\ell \oplus W_\ell$$

orthogonal complement

Fine scale   coarse scale   details

Thm Let $\hat{h}$ be the conj. mirror filter associate to a scaling fct ϕ.

Define $\cancel{\hat{g}(\omega) = e^{i\omega}\hat{h}(\omega)^*}$

$$\hat{g}(\omega) = e^{-i\omega}\hat{h}(\omega+\pi)^*$$

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2}}\hat{g}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right)$$

and $\psi(t) = \int \hat{\psi}(\omega)\, e^{i\omega t}\,\frac{d\omega}{2\pi}$