Ex #2 Piecewise linear

$$\hat{\theta}(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^2 \Rightarrow \ldots$$

Ex #4 Shannon : orthonormal to start with

Conjugate mirror filter

Recall that $V_1 \subseteq V_0$

In particular $\frac{1}{\sqrt{2}}\phi(\cdot/2) \in V_0$

$$\frac{1}{\sqrt{2}}\phi(t/2) = \sum_{k\in\mathbb{Z}} h_k \phi(t-k)$$

Taking Fourier transform

$$\frac{1}{\sqrt{2}}\int \phi(t/2) e^{-i\omega t} dt = \sqrt{2}\,\hat{\phi}(2\omega)$$

$$= \underbrace{\sum_{k\in\mathbb{Z}} h_k e^{-i\omega k}}_{\hat{h}(\omega)} \hat{\phi}(\omega)$$

$$\boxed{\hat{\phi}(2\omega) = \frac{1}{\sqrt{2}} \hat{h}(\omega)\, \hat{\phi}(\omega)}$$

$$h_k = \frac{1}{\sqrt{2}} \langle \phi(\cdot/2), \phi(\cdot - k) \rangle_{L^2}.$$

<u>Thm</u> The following holds

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega+\pi)|^2 = 2 \qquad (1)$$

$$\hat{h}(0) = \sqrt{2}. \qquad (2)$$

Conversely, assume (1) + (2) and

(3) ~~the~~ $\hat{h}$ is 2π-periodic

(4) $\hat{h}$ cont. diff in a neigh of 0

(5) $\inf_{\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]} |\hat{h}(\omega)| > 0$

Then $$\hat{\phi}(\omega) := \prod_{\ell=1}^{\infty} \frac{\hat{h}(2^{-\ell}\omega)}{\sqrt{2}}$$

is the fourier tr. of a scaling function

Wavelets

Note that $V_\ell \subseteq V_{\ell-1}$

So we can decompose

$$V_{\ell-1} = V_\ell \oplus W_\ell$$

orthogonal complement

Fine scale, coarse scale, details

Thm Let $\hat{h}$ be the conj. mirror filter associate to a scaling fct ϕ.

Define $\cancel{\hat{g}(\omega) = e^{i\omega}\hat{h}(\omega)^*}$

$$\hat{g}(\omega) = e^{-i\omega}\hat{h}(\omega+\pi)^*$$

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2}}\hat{g}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right)$$

and $\psi(t) = \int \hat{\psi}(\omega) e^{i\omega t}\frac{d\omega}{2\pi}$

$$\psi_{\ell,k}(t) := \frac{1}{\sqrt{2^\ell}}\,\psi\left(\frac{t-k2^\ell}{2^\ell}\right)$$

Then $\forall \ell$ $\{\psi_{\ell,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of W_ℓ.

<u>Proof</u> sufficient for $\ell=0$.

~~We will check~~ ~~With consi~~

Need to check (0) $\{\psi(\cdot-k)\} \subseteq V_{-1}$

(1) $\{\psi(\cdot-k)\}_{k\in\mathbb{Z}}$ are orthonormal

(2) $\{\psi(\cdot-k)\}_{k\in\mathbb{Z}}$ orthogonal to $\{\phi(\cdot-m)\}_{m\in\mathbb{Z}}$

(3) ~~span~~ $\overline{\text{span}(\{\phi(\cdot-k)\},\{\psi(\cdot-m)\})}$

$$= \overline{\text{span}(\{\phi(2(t-k/2))\})}.$$

(0) is obvious.

~~(4)~~ Will check (1), (2), leave (3) as ~~exercise~~ reading

$\sim \psi_m(\cdot) := \psi(\cdot - m)$

(1) $\langle \psi_m, \psi_k \rangle = \frac{1}{2\pi} \int e^{-i\omega(m-k)} |\hat{\psi}(\omega)|^2 d\omega$

$= \frac{1}{2\pi} \int e^{-i\omega(m-k)} \frac{1}{2} |\hat{g}(\frac{\omega}{2})|^2 |\hat{\phi}(\frac{\omega}{2})|^2 d\omega$

$= \frac{1}{2\pi} \int e^{-2i\omega(m-k)} |\hat{g}(\omega)|^2 |\hat{\phi}(\omega)|^2 d\omega =$

$= \frac{1}{2\pi} \sum_{q \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{-2i\omega(m-k)} |\hat{g}(\omega)|^2 |\hat{\phi}(\omega + 2\pi q)|^2 d\omega$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-2i\omega(m-k)} |\hat{g}(\omega)|^2 d\omega =$

$= \frac{1}{2\pi} \int_{0}^{\pi} e^{-2i\omega(m-k)} \left(|\hat{g}(\omega)|^2 + |\hat{g}(\omega+\pi)|^2 \right) d\omega =$

(*) $= \frac{1}{2\pi} \int_0^{2\pi} e^{-i\omega(m-k)} d\omega = \delta_{m-k}$!

(2)

$$\langle \phi_m, \phi_k \rangle = \oint \frac{1}{2\pi} \int_{\pi} e^{-i\omega(m-k)} \hat{g}\left(\frac{\omega}{2}\right) \hat{h}^*\left(\frac{\omega}{2}\right) \left|\hat{\phi}\left(\frac{\omega}{2}\right)\right|^2 d\omega$$

$$= \frac{1}{2\pi} \sum_{q \in \mathbb{Z}} \int_{-\pi} e^{-2i\omega(m-k)} \hat{g}(\omega) \hat{h}(\omega)^* \left|\hat{\phi}(\omega + 2\pi q)\right|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{g}(\omega) \hat{h}(\omega)^* e^{-2\pi i\omega(m-k)} d\omega$$

$$= \frac{1}{2\pi} \int_{0}^{\pi} \left[\hat{g}(\omega)\hat{h}(\omega)^* + \hat{g}(\omega + \pi)\hat{h}(\omega + \pi)\right] e^{-2i\omega(m-k)} d\omega$$

$$= \cancel{\frac{1}{2\pi}\int_0^\pi [\;]} = 0$$

<u>Claim</u> $\hat{g}(\omega)\hat{h}(\omega)^* + \hat{g}(\omega+\pi)\hat{h}(\omega+\pi)^*$

$$= e^{-i\omega} \hat{h}^*(\omega+\pi)\hat{h}^*(\omega) \cancel{\phi} - e^{-i\omega} \hat{h}(\omega)^* \hat{h}^*(\omega+\pi)$$

$$= 0$$

Example # 1 Piecewise constant.

$$\phi(t) = 1_{[0,1]}(t) \qquad \hat{\phi}(\omega) = \frac{1}{i\omega}(1-e^{-i\omega})$$

$$\frac{1}{\sqrt{2}}\phi\left(\frac{t}{2}\right) = \frac{1}{\sqrt{2}} 1_{[0,2]}(t) = \frac{1}{\sqrt{2}}\phi(t) + \frac{1}{\sqrt{2}}\phi(t-1)$$

$$h_0 = \frac{1}{\sqrt{2}} \qquad h_1 = \frac{1}{\sqrt{2}} \qquad h_k = 0 \ \forall k \notin \{1,2\}$$

$$\hat{h}(\omega) = \frac{1}{\sqrt{2}}(1+e^{-i\omega})$$

$$\hat{g}(\omega) = e^{-i\omega}\frac{1}{\sqrt{2}}(1 + e^{-i\omega})^* = e^{-i\omega}\frac{1}{\sqrt{2}}(1-e^{i\omega})$$

$$= \frac{1}{\sqrt{2}}(e^{-i\omega}-1)$$

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(e^{-i\frac{\omega}{2}}-1)\frac{1}{i\omega}(1-e^{-i\frac{\omega}{2}})$$

$$= \frac{1}{2\omega}(e^{-i\frac{\omega}{2}}-1)^2$$

$$\psi(t) = 1_{[0,\frac{1}{2}]} - 1_{[\frac{1}{2},1]} =$$

$$\psi(t) = c\left(1_{[0,1/2]} - 1_{(1/2,1]}\right)$$

$$= c\left[\frac{e^{-i\frac{\omega}{2}} - 1}{-i\omega} - \frac{e^{-i\omega} - e^{-i\omega/2}}{-i\omega}\right]$$

$$= \frac{ci}{\omega}\left[-1 + 2e^{-i\frac{\omega}{2}} - e^{-i\omega}\right] =$$

$$= \frac{c}{i\omega}\left(1 - e^{-i\omega/2}\right)^2 \qquad c = -1$$

Finally notice that we can iterate the decomposition. Start for instance from $\ell = 0$

$$V_{-1} = V_0 \oplus W_0$$

$$V_{-2} = V_{-1} \oplus W_{-1} = V_0 \oplus W_0 \oplus W_{-1}$$

$$\vdots$$

$$V_{-k-1} = V_0 \oplus \bigoplus_{j=0}^{-k} W_j$$

~~As $\bigcup_{j=1} V_{-j} \to L^2$ as $j \to \infty$~~

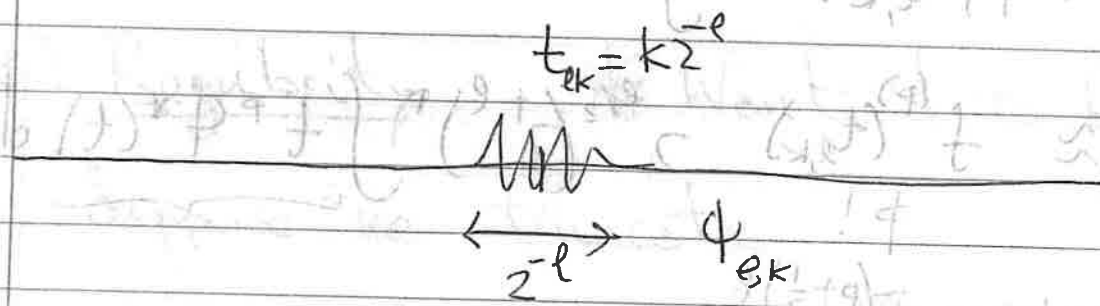
$$\bigcup_{j=1}^{\infty} V_{-j} = L^2,$$ we have

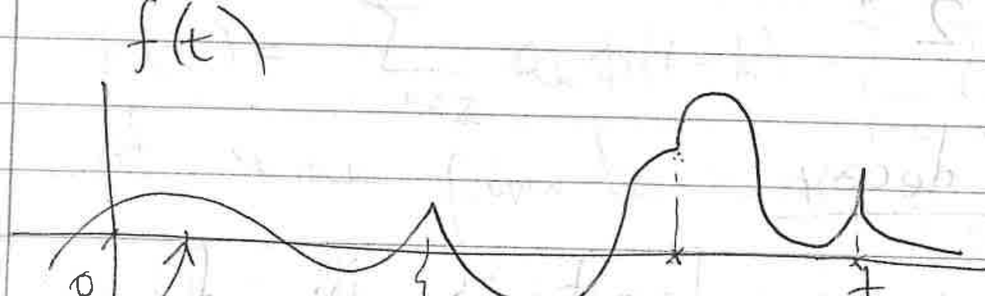
$$L^2(\mathbb{R}) = V_0 \oplus \bigoplus_{\ell=0}^{-\infty} W_\ell$$

In other words $\quad \phi_k(t) := \phi(t-k)$

$$\{\phi_k, \psi_{\ell,k} \quad k \in \mathbb{Z} \quad \ell \leq 0\}$$

Form an orthonormal basis of $L^2(\mathbb{R})$





few nonzero coeff in smooth parts

Wave Many nonzero coef near singularities