Wavelets with additional properties

* Desirable properties.

(1) Vanishing moments

$$\int_{\mathbb{R}} t^k \psi(t)\, dt = 0 \qquad 0 \leq k < p$$

Yields fast decay on smooth functions. Say $f \in C^\infty$.

$$\langle f, \psi_{\ell,k} \rangle = \sqrt{2^{\ell}} \int f(2^{-\ell}k + 2^{-\ell}t)\psi^*(t)\, dt$$

$$\approx \frac{f^{(p)}(t_{\ell,k})}{p!}\, 2^{\ell/2} (2^{-\ell})^p \underbrace{\int t^p \psi^*(t)\, dt}$$

$$\asymp 2^{-(p+\frac{1}{2})\ell}$$

(2) Fast decay

Yiel $\Rightarrow$ singularities in f have only local impact

In particular, ~~if~~ assume $|\underbrace{\mathrm{supp}(\psi)}_{S}| < \bar{K}$

has a singularity at t_0

$\langle f, \psi_{e,j} \rangle$ impacted by t_0 only if

$$\frac{t_0 - 2^{+e} j}{2^{e}} \in S \qquad j \in \frac{t_0}{2^{e}} - S$$

ie there is at most $\bar{K}$ coefficients affected at each scale!

(3) Regularity   Want ψ to be smooth

Suppose we truncate

$$f(t) = \sum_{k \in \mathbb{Z}} a_k \phi(t-k) + \sum_{l=0}^{-\infty} \sum_{k \in \mathbb{Z}} b_{l,k} \psi_{l,k}(t)$$

at some finite sum $\Rightarrow \hat{f}(t)$

- If ϕ, ψ smooth
  then $\hat{f}$ smooth.
- Ex: Haar wavelet is bad in this.

$\forall p \in \mathbb{N}$

<u>Theoren</u> (Daubechies) There exists (ϕ, ψ) (scaling function / wavelet) such that

1) $supp(\psi) = [-p+1, p]$
2) $\int \psi(t) t^k \, dt = 0 \quad \forall\, k < p$
3) $supp(\phi) = [0, 2p-1]$

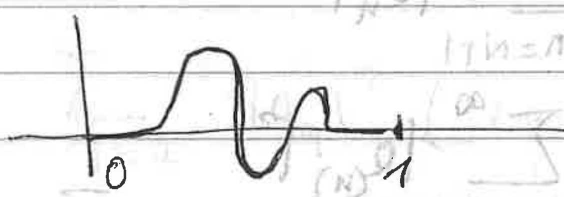
Further there exists no wavelet satisfying (2) with smaller support.

α Holder continuous

	$p=2$	$p=3$	large p
α	0.55	1.08	$0.2p$

Approximation in Fourier / Wavelet bases

Will consider $f : [0,1] \to \mathbb{C}$ with $\mathrm{supp}(f) \subseteq (0,1)$ to avoid boundary effects



~~Linear approx~~ Given any basis φ_n of $L^2([0,1])$ $\quad f(t) = \sum_{n=1}^{\infty} a_n \cancel{f_n}\, \varphi_n(t)$

φ_r

Linear approx

$$\epsilon_L(N, f) := \left\| f - \sum_{n=1}^{N} a_n \varphi_n \right\|_{L^2}^2$$

nonlinear approx

$$\epsilon_N(N, f) := \inf_{\substack{S \subseteq \mathbb{N} \\ |S| \leq N}} \left\| f - \sum_{n \in S} a_n \varphi_n \right\|^2$$

Equivalently if

$a_{(i)}$ are ranked coeff

$|a_{(1)}| \geq |a_{(2)}| \geq \cdots$

$$\epsilon_L(N,f) := \sum_{n=N+1}^{\infty} |a_n|^2$$

$$\epsilon_N(N,f) := \sum_{n=N+1}^{\infty} |a_{(n)}|^2$$

For $\mathcal{F} \subseteq L^2([0,1])$

$$\epsilon_{L/N}(N,\mathcal{F}) := \sup\{\epsilon_{L/N}(N,f) : f \in \mathcal{F}\}$$

~~For instance, we~~

Note that this depends on the basis

For instance we saw that

Fourier :

$$\epsilon_L^F(N, W^{s,2}(R)) \lesssim R^2 N^{-2s}$$

Indeed $\boxed{f \in W^{s,2}(R)}$ iff

$$\sum_{q \in \mathbb{Z}} |a_q|^2 (1 + |2\pi q|^{2s}) \le R^2$$

$\cancel{|a_{N/2}|}$

$\geq$ $\qquad$ a

$$(1 + 2|2\pi \tfrac{N}{2}|^{2s}) \sum_{|q| > \frac{N}{2}} |a_q|^2 = \epsilon(1+$$

$$\geq C \cdot \epsilon(f, N/2)\, N^{2s}$$

In fact it is easy to see that this bound is tight

$$\epsilon_L(N, W^{s,2}(R)) = R^2 N^{-2s+o(1)}$$

$$\epsilon_N(N, W^{s,2}(R)) = R^2 N^{-2s+o(1)}.$$

Further the same rates are obtained by Wavelet bases.

A different example.

$\boxed{\alpha \geq 1/2}$

$f \in C^\alpha$ except at k discontinuities

$\mathcal{F}_{\alpha,k}(R)$

$\mathcal{F}_{\alpha,k}(R) := \{ f : \exists\, t_1 \ldots t_k$ st

$\forall \ell\ \forall t_* \in [t_{\ell-1}, t_\ell)$ $\exists$ polynomial q_* of deg $\lfloor \alpha \rfloor$ such that

$\forall t \in (t_{\ell-1}, t_\ell)$ $|f(t) - q_*(t)| \leq R \cdot |t - t_*|^{\alpha} \}$

$\|f\|_\infty \leq R$

Theorem: Assume a wavelet basis with bounded support and $q > \alpha$ vanishing moments. Then

$$\epsilon_L(N, \mathcal{F}_{\alpha K}(R)) \leq C R^2 \cdot N^{-1} \qquad (*)$$

$$\epsilon_N(N, \mathcal{F}_{\alpha K}(R)) \leq C R^2 N^{-2\alpha} \qquad (**)$$

Proof idea

$$f(t) = \sum_{k=-M}^{\overset{\sim O(1)}{M}} c_k \phi(t-k) + \sum_{\ell=-1}^{\infty} \sum_{k=-M}^{M+2^{-\ell}} a_{\ell,k} \psi_{\ell,k}(t)$$

M is some finite fixed number so we can neglect the coefficient c_k and some of the $a_{\ell,k}$. Assume wlog

$$f(t) = \sum_{\ell=-1}^{\infty} \sum_{k=1}^{2^{-\ell}} a_{\ell,k} \psi_{\ell,k}(t).$$

$$\cancel{R^2 \geq \|f\|_{L^2}^2 = \sum_{\ell=-1}^{\infty} \|a_{\ell,\cdot}\|_2^2}$$

Two types of coefficients

$(\ell,k) \in S_1 \iff \mathrm{supp}(\psi_{\ell,k}) \cap \{t_1 \dots t_K\} \neq \emptyset$

$$\overset{\infty}{\underset{\ell=-1}{\bigcup}} S_1^{(\ell)} \qquad |S_1^{(\ell)}| \leq C \quad \forall \ell$$

$$(\ell,k) \in S_2^{t} = \overset{-\infty}{\underset{\ell=-1}{\bigcup}} S_2^{(\ell)} \text{ otherwise}$$

$$(\ell,k) \in S_1^{(\ell)} \Rightarrow \cancel{a_{\ell,k} = \int \langle \psi_{\ell,k},}$$

$$|a_{\ell,k}| = \left| \int \psi_{\ell,k}(t)^* f(t)\, dt \right| \leq R 2^{+\frac{\ell}{2}}$$

$\cancel{\ell}$

$$\frac{t-k2^\ell}{2^\ell} = u$$

$$(\ell,k) \in S_2^{(\ell)} \Rightarrow d$$

$$\cancel{|a_{\ell,k}| = \left| \int \psi(t + 2^\ell)\right.} f(t+k2^\ell)$$

$$|a_{\ell,k}| = \left| \int \psi(t)^* f(k2^\ell + 2^\ell t)\; 2^{\ell/2}\, dt \right|$$

$\leq CR\, 2^{\ell(\alpha+\frac{1}{2})}$

$\|a_{\ell,\cdot}\|_2^2 \leq C\left[R^2 2^{\ell} + R^2 2^{\ell(2\alpha+1)} \cdot 2^{-\ell}\right]$

this always dominates $\alpha \geq \frac{1}{2}$

$\epsilon_L(2^L, f) \leq CR^2 \sum_{\ell=L+1}^{\infty} 2^{-\ell} = CR^2 2^{-L}$

$$\boxed{\epsilon_L(N, f) \lesssim R^2 N^{-1}}$$

Now consider nonlinear approx.

Will select ~~$\Lambda_f = \{(\ell,k) : |a_{\ell,k}| \geq \}$~~

- ~~Al~~ $N/2$ top coeff of type $S_1 \geq T_1$
- $N/2$ top coeff of type $S_2 \geq T_2$

$$\epsilon_N(N,f) \leq \underbrace{\sum_{(\ell,k)\in S_1\setminus T_1} |a_{\ell,k}|^2}_{\epsilon^{(1)}} + \underbrace{\sum_{(\ell,k)\in S_2\setminus T_2} |a_{\ell k}|^2}_{\epsilon^{(2)}}$$

error by using top $\ell_0 = \frac{N}{C}$ scales

$$\epsilon^{(1)} \leq \sum_{\substack{(\ell,k)\in S_1 \\ \ell \leq \frac{N}{C}\ \cancel{|\ell| \leq \frac{N}{C}}}} |a_{\ell,k}|^2 \leq C \sum_{\ell=-\frac{N}{C}}^{\infty} R^2 2^{\ell}$$

$$= R^2 2^{-\frac{N}{C}} \quad \text{very small!}$$

$$\epsilon^{(2)} \leq \sum_{\substack{(\ell,k)\in S_2 \\ \ell\leq 0 \quad 2^{-\ell} \leq \frac{N}{16}}} |a_{\ell,k}|^2 \leq$$

$$\leq CR^2 \sum_{\substack{\ell\leq 0 \\ 2^{-\ell} \leq \frac{N}{16}}} 2^{2\alpha\ell} \leq \underline{CR^2 N^{-2\alpha}}$$

$\hookrightarrow$ determines the rate!

A different example: Bounded variation functions : $f:[0,1]\to\mathbb{C}$

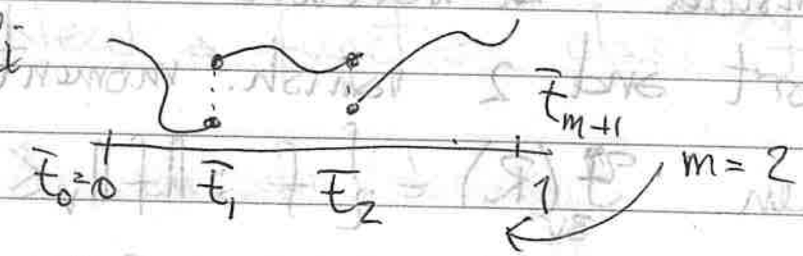
$$\|f\|_V := \sup_N \sup_{0=t_0<t_1 \dots \le t_N=1} \sum_{i=1}^{N} |f(t_i)-f(t_{i-1})|$$

f has ~~fin~~ bounded variation if $\|f\|_V<\infty$

If f differentiable, then

$$\|f\|_V = \int_{[0,1]} |f'(t)|\,dt$$

Eg:



If f diff + jumps at $\bar t_1, \bar t_2 \dots \bar t_m$

$$\|f\|_V = \sum_{\ell=1}^{m} |f(\bar t_\ell^{+})-f(\bar t_\ell^{-})| + \int$$

$$+ \sum_{\ell=0}^{m} \int_{\bar t_\ell}^{\bar t_{\ell+1}} |f'(t)|\,dt$$

$\|f\|_V < \infty$ iff

$$f(t) = \int_{[0,t]} \mu(dt) \quad \text{a. every } t$$

measure valued deriv.

with ~~the~~ μ a complex valued measure

$$\mu(dt) = \nu(dt) \cdot e^{i\phi(t)}$$

measure

$$\|f\|_V = \int_{[0,1]} \nu([0,1])$$

Thm ~~Consider~~ Consider ~~the~~ wavelets with bounded support and 2 vanish. moments.

Then $\mathcal{F}_{BV}(R) = \{ f : \|f\|_V \le R \}$

$$\epsilon_L(N, BV(R)) \le C \cdot R^2 N^{-1}$$

$$\epsilon_N(N, BV(R)) \le C R^2 N^{-2}$$