EE378B Homework 7 Solution

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1 Part (a)

Direct application of Wielandt-Hoffman inequality gives

$$\sum_{i=1}^{n} \left| \lambda_i \left( W_n \sqrt{n} \right) - \lambda_i \left( W^K_n \sqrt{n} \right) \right|^2 \leq \frac{1}{n} \left\| W_n - W^K_n \right\|_F^2 = \frac{1}{n} \sum_{i \neq j} \xi_{ij}, \quad (1)$$

where $\xi_{ij}$ are i.i.d. random variables having the same distribution as $W_{12}^2 1_{|W_{12}| > K}$. Therefore by SLLN with probability 1,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left| \lambda_i \left( W_n \sqrt{n} \right) - \lambda_i \left( W^K_n \sqrt{n} \right) \right|^2 \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{i \neq j} \xi_{ij} = \lim_{n \to \infty} n \cdot \frac{1}{n(n-1)} \sum_{i \neq j} \xi_{ij} = \mathbb{E} \left[ W_{12}^2 1_{|W_{12}| > K} \right]. \quad (2)$$

2 Part (b)

For simplicity we write

$$\frac{1}{n} \sum_{i=1}^{n} \left| \lambda_i \left( W_n \sqrt{n} \right) - \lambda_i \left( W^K_n \sqrt{n} \right) \right|^2 = \Delta_{n,K}. \quad (3)$$

As a consequence of part (a) we know $\limsup_{n \to \infty} \Delta_{n,K} \leq \Delta_K$. By Markov’s inequality

$$\frac{1}{n} \# \left\{ i : \left| \lambda_i \left( W_n \sqrt{n} \right) - \lambda_i \left( W^K_n \sqrt{n} \right) \right| \geq \epsilon \right\} \leq \frac{\Delta_{n,K}}{\epsilon^2}, \quad (4)$$

which allows us to derive that

$$F^K_n(x - \epsilon) - F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \left( 1_{\lambda_i(W^K_n \sqrt{n}) \leq x - \epsilon} - 1_{\lambda_i(W_n \sqrt{n}) \leq x - \epsilon} \right)$$

$$\leq \frac{1}{n} \sum_{i=1}^{n} 1_{\lambda_i(W^K_n \sqrt{n}) \leq x - \epsilon, \lambda_i(W_n \sqrt{n}) > x}$$

$$\leq \frac{1}{n} \# \left\{ i : \left| \lambda_i \left( W_n \sqrt{n} \right) - \lambda_i \left( W^K_n \sqrt{n} \right) \right| \right\}$$

$$\leq \frac{\Delta_{n,K}}{\epsilon^2}. \quad (5)$$

Similarly we bound $F_n(x) - F^K_n(x + \epsilon) \leq \frac{\Delta_{n,K}}{\epsilon^2}$. For the above two inequalities, take lim sup on both sides and rearrange terms, which gives the desired result

$$F^K(x - \epsilon) - \frac{\Delta_K}{\epsilon^2} \leq \liminf_{n \to \infty} F_n(x) \leq \limsup_{n \to \infty} F_n(x) \leq F^K(x + \epsilon) + \frac{\Delta_K}{\epsilon^2}. \quad (6)$$
Finally, if $F^K(x) \to F(x)$ for every $x \in \mathbb{R}$ as $K \to \infty$, using the fact that $\Delta_K \to 0$ since $W_{12}$ has finite second moment, we obtain by taking $K \to \infty$ that

$$F(x - \epsilon) \leq \liminf_{n \to \infty} F_n(x) \leq \limsup_{n \to \infty} F_n(x) \leq F(x + \epsilon).$$

(7)

Whenever $F$ is continuous at $x$, by taking $\epsilon \to 0$ it is concluded that $\lim_{n \to \infty} F_n(x) = F(x)$.

3 Part (c)

In fact, it suffices to show that for all $K$ with $\beta_K = \text{Var} \left(W_{12} 1_{|W_{12}| \leq K}\right)$,

$$F_n^K(x) \to F^K(x) := \frac{1}{2\beta_K \pi} \int_{-\infty}^x \sqrt{4\beta_K - t^2} \mathbb{1}_{|t| \leq 2\sqrt{\beta_K}} \text{d}t,$$

i.e., the semi-circular law with variance $\beta_K$. Then all conditions hold since $\beta_K \to 1$ as $K \to \infty$ and $F^K(x) \to F(x)$ pointwise with $F(x)$ being the semi-circular law CDF with variance 1.

Let $\tilde{W}_n^K = W_n^K + \alpha_K 11^\top$ where $\alpha_K = -\mathbb{E}[W_{12} 1_{|W_{12}| \leq K}]$. Then $\mathbb{E}[\tilde{W}_n^K] = 0$ and we can apply results that have been shown in class that

$$\tilde{F}_n^K(x) \to F^K(x),$$

(9)

where $\tilde{F}_n^K(x)$ is the empirical CDF of eigenvalues of $\tilde{W}_n^K / \sqrt{n}$. The proof will be concluded once we can establish

$$\left| \tilde{F}_n^K(x) - F_n^K(x) \right| \to 0$$

(10)

for any fixed $K$ and $x \in \mathbb{R}$. By the generalized Weyl's inequality, we're able to get

$$\lambda_i(\tilde{W}_n^K / \sqrt{n}) \geq \lambda_{i+1}(\tilde{W}_n^K / \sqrt{n}) + \lambda_{n-1}(\alpha_K 11^\top / \sqrt{n}), \quad i = 1, 2, \cdots, n-1;$$

(11)

$$\lambda_i(\tilde{W}_n^K / \sqrt{n}) \leq \lambda_{i-1}(\tilde{W}_n^K / \sqrt{n}) + \lambda_2(\alpha_K 11^\top / \sqrt{n}), \quad i = 2, 3, \cdots, n.$$  

(12)

While $\alpha_K 11^\top / \sqrt{n}$ is only a rank-1 matrix, it follows when $n \geq 3$ that

$$\lambda_2(\alpha_K 11^\top / \sqrt{n}) = \lambda_{n-1}(\alpha_K 11^\top / \sqrt{n}) = 0,$$

(13)

which gives us

$$\lambda_i(\tilde{W}_n^K / \sqrt{n}) \geq \lambda_{i+1}(W_n^K / \sqrt{n}), \quad i = 1, 2, \cdots, n-1;$$

(14)

$$\lambda_i(\tilde{W}_n^K / \sqrt{n}) \leq \lambda_{i-1}(W_n^K / \sqrt{n}), \quad i = 2, 3, \cdots, n.$$  

(15)

By the first bound, one can deduce that

$$F_n^K(x) - F_n^K(x) = \frac{1}{n} \sum_{i=1}^n \left( \mathbb{1}_{\lambda_i(\tilde{W}_n^K / \sqrt{n}) \leq x} - \mathbb{1}_{\lambda_i(W_n^K / \sqrt{n}) \leq x} \right) \leq \frac{1}{n} \sum_{i=1}^{n-1} \left( \mathbb{1}_{\lambda_i(\tilde{W}_n^K / \sqrt{n}) \leq x} - \mathbb{1}_{\lambda_{i+1}(W_n^K / \sqrt{n}) \leq x} \right) + \frac{1}{n} \leq \frac{1}{n}.$$  

(16)

The other side is similar, and therefore $\left| F_n^K(x) - F_n^K(x) \right| \leq 1/n \to 0$. The proof is done.
4 Part (d)

We can generate uniformly distributed (in the sense of Haar’s measure) orthogonal matrices $U \in \mathbb{R}^{n \times k}$ by first generate an $n$ by $k$ matrix with i.i.d. standard Gaussian entries and perform Gram-Schmidt orthogonalization. Since the distribution of Gaussian matrix is invariant under orthogonal transformation, the distribution of resulting orthogonal matrix $U$ is also invariant, and thus is uniformly distributed under Haar’s measure.

(i) The plots are shown in Figure 1.

(ii) The result is different from Winger’s semicircle’s law whereas the histogram should look like a semicircle.

(iii) We can’t apply since the entries $Y_{ij}$ are correlated.

Remark from class: What happens for a larger choice of constant? Does it recover the semicircle law?