This is a mock final: it is analogous to the actual final in length and difficulty but not necessarily in structure.

The objective of this final is to study the locally rigid embedding (LRE) method proposed in


This is a positioning algorithm. Given a graph $G = (V, E)$ over $n = |V|$ vertices and distance measurements $d_{ij}$ for $(i, j) \in E$, the algorithm tries to find positions $(r_1, \ldots, r_n)$ in $p$ dimensions such that $\|r_i - r_j\|_2 \approx d_{ij}$ for each $(i, j) \in E$. Throughout this exercise we assume that the dimension is $p = 2$, and let $r_i = (x_i, y_i)$. Further, we let $x = (x_1, x_2, \ldots, x_n)^* \in C$, $y = (y_1, y_2, \ldots, y_n)^*$, and $u = (1, 1, \ldots, 1)^*$.

Here is a synthetic description of the algorithm. For further details we refer to the original paper (posted on the class webpage).

The idea is to construct a symmetric positive semidefinite matrix $S \in \mathbb{R}^{n \times n}$, $S \succeq 0$, such that

$$Su = 0, \quad Sx = 0, \quad Sy = 0. \quad (1)$$

while $\langle v, Sv \rangle > 0$ strictly whenever $v$ is orthogonal to $u, x, y$.

In order to construct this matrix, let $C_1, \ldots, C_n$ be a collection of cliques in $G$, whereby $C_i$ includes $i$ and its $k_i$ closest neighbors (for $k_i > p + 1 = 3$ suitably chosen). For each clique $C_i$, find the positions $\{r_j\}_{j \in C_i}$ (up to a rigid motion) using multi-dimensional (MDS). Let $u_{C_i} \in \mathbb{R}^{k_i+1}$ be the all-one vector $u_{C_i} = (1, 1, 1, \ldots, 1)$, and define $x_{C_i} = (x_j)_{j \in C_i}$ and $y_{C_i} = (y_j)_{j \in C_i}$.

Then construct a symmetric positive matrix $P_{C_i} \in \mathbb{R}^{(k_i+1) \times (k_i+1)}$ with $P_{C_i} u_{C_i} = P_{C_i} x_{C_i} = P_{C_i} y_{C_i} = 0$. More precisely, $P_{C_i}$ is the projector orthogonal to $u_{C_i}, x_{C_i}, y_{C_i}$. This is obtained by constructing (e.g. via Gram-Schmidt orthogonalization) $k_i - 2$ orthonormal vectors $w_{i,1}, \ldots, w_{i,k_i-2}$, that are orthogonal to $u_{C_i}, x_{C_i}, y_{C_i}$, and letting

$$P_{C_i} = \sum_{\ell=1}^{k_i-2} w_{i,\ell} w_{i,\ell}^T. \quad (2)$$

Next, let $\tilde{P}_{C_i} \in \mathbb{R}^{n \times n}$ be the matrix that is obtained from $P_{C_i}$ by letting $(\tilde{P}_{C_i})_{jl} = 0$ if $j \not\in C_i$ or $l \not\in C_i$, and filling the remaining entries using the corresponding entries of $P_{C_i}$. In terms of these we let

$$S = \sum_{i=1}^n \tilde{P}_{C_i}.$$

Finally, the positions $(r_1, \ldots, r_n)$ are reconstructed by using the 3 eigenvectors of $S$ corresponding to the smallest eigenvalues. One of this is always $u$. The other two yield $x, y$ (up to an affine transformation).

(1) Prove that the matrix $S$ constructed above is positive semidefinite. Prove that $Su = 0$ and that (if the distance measurements $d_{ij}$ are exact) $Sx = Sy = 0$.

(2) Implement the LRE algorithm. Describe your choice of $k_i$ and justify it.
(3) Test your implementation on synthetic data. More precisely, generate $n = 1000$ i.i.d. positions $r_1, \ldots, r_n$ uniformly at random in the unit square $[-1/2, 1/2] \times [-1/2, 1/2]$. Generate the random geometric graph $G$ by letting $(i, j) \in E$ if $\|r_i - r_j\|_2 \leq \rho$. Run the LRE algorithm for several values of $\rho$ and report your results.

(4) Comment on the algorithm complexity.