Logistics

- All the course-related information regarding grading, office hours, handouts, announcements, and class project: this is required for 384Y is posted on the class website.

- Please take a look at the website right away, esp re project deadlines.
  - some past projects and project reports are also posted.

- Please subscribe to the class mailing list.

- Note: the class won’t be on SITN (instruction is live).
Overview of Topics

- Review
- Stability of switch scheduling algorithms
- Stability of the Max Weight Matching Algorithm
Review of stability

- Last quarter we saw
  - HoL blocking limited throughput in IQ switches to \(2 - \sqrt{2} \approx 58\%\)
  - the use of VOQs removes HoL blocking, making it possible to get higher throughputs
  - the definition of the stability of Markov chains
  - how to use Foster’s criterion to prove stability

- Stability of switch scheduling algorithms
  - when the arrival matrix, \(\Lambda = \{\lambda_{ij}\}\), is known:
    - use the Birkhoff-von Neumann decomposition to get a schedule
  - when \(\Lambda\) is unknown, use \(Q(t) = \{q_{ij}(t)\}\), the queue-sizes
  - we saw that the Max Size Matching was unstable, but the Max Weight Matching could be stable
  - the proof of MWM’s stability involved using Foster’s criterion and a quadratic Lyapunov function
• Foster’s criterion

- Let $X_n$ be a discrete-time Markov chain on the countable state space $S$. Suppose that it is aperiodic and irreducible.

- Let $L : S \rightarrow \mathbb{R}^+$ be a Lyapunov function, and let $C = \{x \in S : L(x) \leq B\}$; i.e. $L(x)$ is bounded in $C$.

• Stability theorem:

(a) If there exists an $\epsilon > 0$ and $B_1 > 0$ such that $E[L(X_{n+1}) - L(X_n) | X_n \notin C] \leq -\epsilon$ and

\[ \sup\{L(X_{n+1}) - L(X_n) : X_n \in C\} \leq B_1, \]

then

\[ \limsup_{n \to \infty} L(X_n) < \infty \] with probability 1.

(b) Further, if there exits a monotonically increasing positive function $f$ such that

\[ E[L(X_{n+1}) - L(X_n) | X_n \notin C] \leq -\epsilon f(L(X_n)) \]

then

\[ \limsup_{n \to \infty} E[f(L(X_n))] < \infty. \]


Stability of the MWM algorithm

- Notation: consider an $N \times N$ IQ switch with VOQs
  - arrivals form a Bernoulli IID process and are denoted by $\{A_{ij}(t)\}$
  - the arrival rate vector: $\lambda = \{\lambda_{ij}\}$
    (note: we’re thinking of $\lambda$ as a $N^2 \times 1$ column vector)
  - let $S(t) = \{S_{ij}(t)\}$ denote the “service vector” or “matching” used at $t$
    - $S_{ij}(t) = 1$ iff input $i$ is connected to output $j$ at time $t$
    - $S(t)$ has exactly one 1 for each input $i$ and output $j$
  - the queue-size vector at time $t$ evolves according to:
    $$q(t + 1) = [q(t) - S(t)]^+ + A(t + 1)$$  \hspace{1cm} (1)
  - let $W(t) = \sum_{ij} q_{ij}(t)S_{ij}(t) = \langle q(t), S(t) \rangle$ be the weight of the matching used at time $t$
  - given $q(t)$, the Maximum Weight Matching algorithm uses the service vector
    $$S^*(t) = \arg \max_S \langle q(t), S \rangle$$
    with the biggest weight
  - let $W^*(t) = \langle q(t), S^*(t) \rangle$ be the weight of the MWM at time $t$
- Lyapunov analysis
  - consider a quadratic Lyapunov function: \( L(t) = \sum_{ij} q_{ij}^2(t) = \langle q(t), q(t) \rangle \)
  - we need to show: there is some \( \epsilon > 0 \) and some \( K > 0 \) such that
    \[
    E[L(t + 1) - L(t) | q(t)] \leq -\epsilon W^*(t), \quad \text{whenever} \quad W^*(t) \geq K \tag{2}
    \]

- Consider
    \[
    L(t + 1) - L(t) = \sum_{ij} q_{ij}^2(t + 1) - q_{ij}^2(t) = \sum_{ij} (q_{ij}(t + 1) - q_{ij}(t)) (q_{ij}(t + 1) + q_{ij}(t))
    \]

- Note that for each \( i, j \) and for every \( t \)
    \[
    q_{ij}(t + 1) - q_{ij}(t) \leq \begin{cases} 
    1 & \text{if } q_{ij}(t) = 0 \\
    A_{ij}(t + 1) - S_{ij}(t) & \text{otherwise}
    \end{cases}
    \]

- Therefore
    \[
    L(t + 1) - L(t) \leq \sum_{ij} (A_{ij}(t + 1) - S_{ij}(t)) (2q_{ij}(t) + 1) + 1
    \leq \sum_{ij} (A_{ij}(t + 1) - S_{ij}(t)) (2q_{ij}(t)) + 2N^2
    \]
• Taking conditional expectations

\[ E[L(t + 1) - L(t)|q(t)] \leq 2 \sum_{ij} q_{ij}(t) [E(A_{ij}(t + 1) - S_{ij}(t)|q(t)] + 2N^2 \]

\[ = 2 \sum_{ij} q_{ij}(t) [\lambda_{ij} - E(S_{ij}(t)|q(t))] + 2N^2 \]

• Now

- because our switch is using the MWM, it follows that \( E(S(t)|q(t)) = S^*(t) \)
- and that \( \sum_{ij} q_{ij}(t) E(S_{ij}(t)|q(t)) = \langle q(t), S^*(t) \rangle = W^*(t) \)
- because \( \lambda \) is admissible, it is strictly doubly-stochastic
- therefore we can write \( \lambda = \sum_k \alpha_k P_k \), where \( P_k \) are permutations and \( \sum_k \alpha_k = \alpha < 1 \)
- therefore,

\[ \langle \lambda, q(t) \rangle = \sum_k \alpha_k \langle P_k, q(t) \rangle \]

\[ \leq \sum_k \alpha_k W^*(t) = \alpha W^*(t) \]

• Using this, we get the stability of MWM as follows:

\[ E[L(t + 1) - L(t)|q(t)] \leq 2\langle \lambda, q(t) \rangle - W^*(t) + 2N^2 \]

\[ = -\delta W^*(t) + 2N^2, \text{ where } \delta = 2(\alpha - 1) \]
Consequences, variants

• So we finally have an “online” algorithm which is stable
  - it doesn’t require a knowledge of $\Lambda$, it doesn’t estimate it
  - it just works off of the queue-size (a good “sufficient statistic” for the arrivals)
  - this is pretty much the only algorithm that had been known to be stable at speedup 1
  - i.e.; every other algorithm and proof was essentially a variant of MWM
  - this included some interesting randomized algorithms (see next lecture)
    → in the next class you will see some recent work which pursues a new direction: a stable algorithm that is based on randomized edge-coloring

• Some of the early variants are quite notable
  - Oldest Cell First
    - the weight of the edge $(i, j)$ equals the age of the HoL cell in $VOQ_{ij}$
    - you will see this in a homework exercise
  - Longest Port First: this algorithm is due to Mekkitikkul and McKeown
• The LPF algorithm
  - let \( \{q_{ij}(t)\} \) be the queue-sizes
  - let \( r_i(t) = \sum_j q_{ij}(t) \) be all the packets at input \( i \)
  - let \( c_i(t) = \sum_i q_{ij}(t) \) be all the packets destined for output \( j \)
  - let \( w_{ij}(t) \) be the weight of the edge \((i, j)\), where
    \[
    w_{ij}(t) = \begin{cases} 
    r_i(t) + c_j(t) & \text{if } q_{ij}(t) > 0 \\
    0 & \text{otherwise}
    \end{cases}
    \]

• The LPF algorithm finds the Max Wt Matching in the bipartite graph whose edges have weight \( \{w_{ij}(t)\} \)
  - key property: the LPF algorithm finds the maximum weighted, maximum size matching
  - this is key because it makes LPF be delay-optimal in a certain set of MWM algorithms
  - the above is paradoxical: recall that the MSM is not stable
  - however, LPF which finds the max weighted MSM is not only stable, but delay-optimal!
• Stability of LPF: uses a simple trick

- let $T$ be the $N^2 \times N^2$ positive definite matrix defined as follows

$$T_{ij} = \begin{cases} 
2 & \text{if } i = j \\
1 & \text{if } \lfloor \frac{i}{N} \rfloor = \lfloor \frac{j}{N} \rfloor \\
1 & \text{if } i \mod N = j \mod N \\
0 & \text{else}
\end{cases}$$

- basically, all that matters about $T$ to us is this: $w(t) = Tq(t)$

- because the Lyapunov function $L(t) = \langle q(t), w(t) \rangle = \langle q(t), Tq(t) \rangle$

  can be used prove the stability of LPF exactly as the stability of MWM

→ we revisit this argument when we do stability proofs via fluid models