Alternating Projections

- alternating projection algorithm
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Alternating projection algorithm

$C, D$ closed convex sets in $\mathbb{R}^n$; goal is to find point in $C \cap D$

let $P_C, P_D$ denote projection onto $C$ and $D$

• start with any $x_0 \in C$
• alternately project onto $C$ and $D$:

$$y_k = P_D(x_k), \quad x_{k+1} = P_C(y_k)$$

generates sequence $x_k \in C, y_k \in D$
first few iterations for case \( C \cap D \neq \emptyset \): 

\[
\begin{aligned}
&\text{. . . suggests } x_k, y_k \text{ converge to a point } x^* \in C \cap D \\
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\end{aligned}
\]
first few iterations for case $C \cap D = \emptyset$:

... suggests $x_k \to x^*$, $y_k \to y^*$, with $\|x^* - y^*\|_2 = \text{dist}(C, D)$
Convergence results

• if $C \cap D \neq \emptyset$, $x_k$ and $y_k$ both converge to a point $x^* \in C \cap D$ (Cheney and Goldstein, 1959)

• if $C \cap D = \emptyset$ and $\text{dist}(C, D)$ is achieved, then $x_k \to x^* \in C$, $y_k \to y^* \in D$, where $\|x^* - y^*\|_2 = \text{dist}(C, D)$

many generalizations, e.g., sequential projection onto $k > 2$ sets, . . .
Example: Positive semidefinite matrix completion

- some entries of matrix in $\mathbb{S}^n$ fixed; find values for others so completed matrix is PSD

- $C = \mathbb{S}_+^n$, $D$ is (affine) set in $\mathbb{S}^n$ with specified fixed entries

- projection onto $C$ by eigenvalue decomposition, truncation: if $Y_k = \sum_{i=1}^{n} \lambda_i q_i q_i^T$,

\[
P_C(Y_k) = \sum_{i=1}^{n} \max\{0, \lambda_i\} q_i q_i^T
\]

- projection of $X_k$ onto $D$ by re-setting specified entries to fixed values
specific example:

\[
X = \begin{bmatrix}
4 & 3 & ? & 2 \\
3 & 4 & 3 & ? \\
? & 3 & 4 & 3 \\
2 & ? & 3 & 4 \\
\end{bmatrix}
\]

- initialize with \( Y_0 = X \), with ? entries set to 0
- \( Y_k \) have correct fixed entries; \( X_k \) are PSD
- \( d_k = \| X_k - Y_{k-1} \|_F \) is distance from \( Y_{k-1} \) to PSD cone
- \( \tilde{d}_k = \| Y_k - X_k \|_F \) is norm of error in fixed entries
convergence is linear:

\[ d_k, \tilde{d}_k \]
Relaxation method for linear inequalities

(Agmon 1954) find a point in (non-empty) polyhedron

\[ P = \{ x \mid a_i^T x \leq b_i, \ i = 1, \ldots, m \} \]

use sequential projection onto the \( m \) halfspaces \( a_i^T x \leq b_i \)

- projection of \( z \) onto halfspace \( a_i^T x \leq b_i \) is

\[
P_i(z) = \begin{cases} 
  z & a_i^T z \leq b_i \\
  z - (a_i^T z - b_i) a_i & a_i^T z > b_i 
\end{cases}
\]

- cycle through these projections to find point in \( P \)
Example

find point in $\mathcal{P} \subseteq \mathbb{R}^{100}$, $m = 1000$ inequalities, $\|a_i\|_2 = 1$

maximum constraint violation: $r_k = \max_{i = 1, \ldots, m} \max \{0, a_i^T x_k - b_i\}$