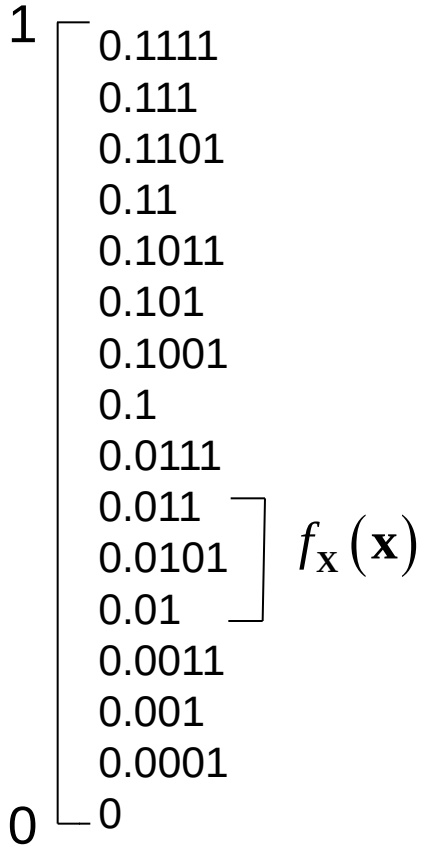
Arithmetic coding

- Elias coding
- Arithmetic coding as finite-precision Elias coding
- Redundancy due to finite precision
- Multiplication-free arithmetic coding
- Context-adaptive coding

Elias coding

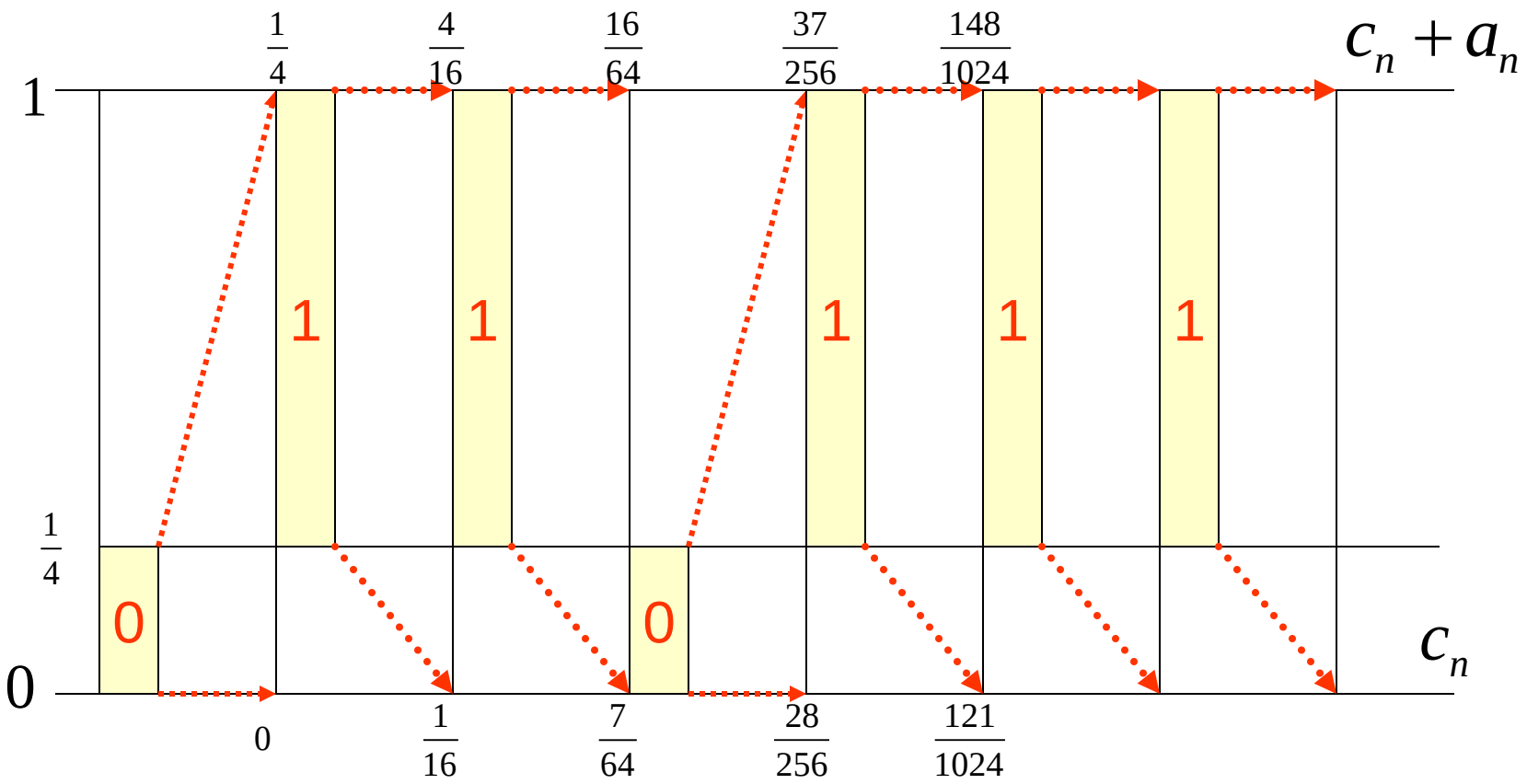
- Entropy coding algorithm for sequences of symbols $\mathbf{x}$ with general (conditional) probabilities
- Representation of $\mathbf{x}$ by a subinterval of the unit interval $[0,1)$
- Width of the subinterval is approximately equal to the probability $f_{\mathrm{X}}(\mathbf{x})$
- Subinterval for $\mathbf{x}$ can be determined by recursive subdivision algorithm
- Represent $\mathbf{x}$ by shortest binary fraction in the subinterval
- Subinterval of width $f_{\mathrm{X}}(\mathbf{x})$ is guaranteed to contain one number that can be represented by L binary digits, with

$$L \approx -\log_2 f_{\mathrm{X}}(\mathbf{x})$$

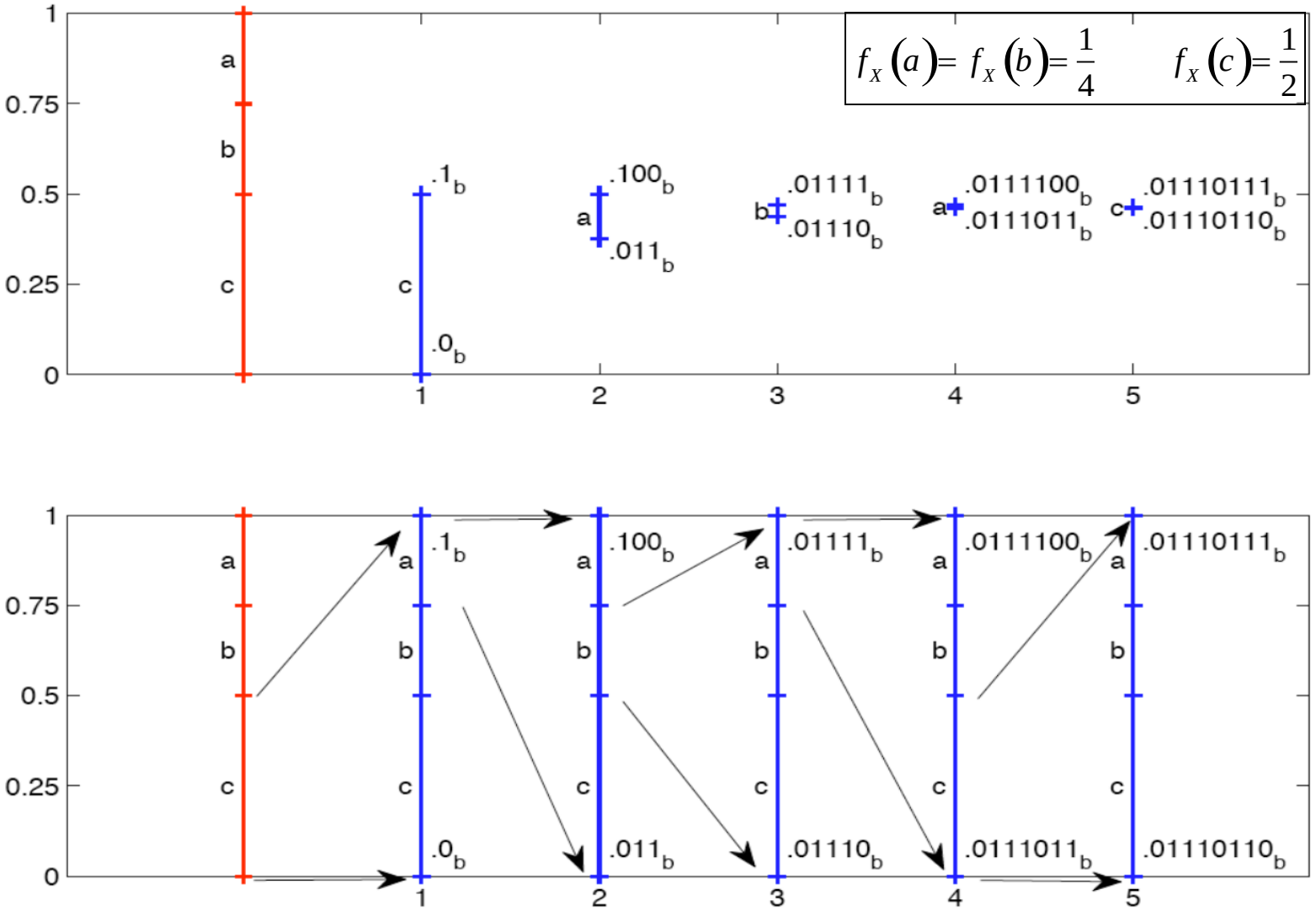


Example: Elias coding of memoryless binary source

$$f_X(0) = \frac{1}{4} \qquad f_X(1) = \frac{3}{4}$$



Example: Elias coding of memoryless ternary source



Courtesy: T. Wiegand

Elias coding: choose binary fraction in subinterval

- Uniquely identify interval $[c_n, c_n + a_n)$ by a binary fraction

$$0.\underbrace{bbbbbb\ldots b}_{L_n \text{ bits}} \in [c_n, c_n + a_n)$$

- Length of bit-string

$$L_n = \left\lceil \log_2 \frac{1}{a_n} \right\rceil + 1 = \left\lceil h_{\mathbf{x}_{0:n-1}}\left(\mathbf{x}_{0:n-1}\right) \right\rceil + 1$$

- Bit-string

$$\hat{c}_n = 2^{-L_n} \left\lfloor 2^{L_n} c_n + 1 \right\rfloor > c_n$$

Successive decoding

- Elias bit string can be decoded symbol by symbol, starting with the first symbol
- For each symbol $n=0,1,2,\ldots$
  - Calculate the intervals $[c_n, c_n + a_n)$ corresponding to all possible x_n
  - Determine the interval for which $\hat{c}_n \in \left[c_n, c_n + a_n \right)$
  - Emit the corresponding x_n

Elias coding for memoryless source

Elias coding algorithm

Start with unit interval

Initialize $c_0 = 0$ and $a_0 = 1$.

For each $n = 0, 1, \ldots$

Update $a_{n+1} \leftarrow a_n f_X(x_n)$

Update $c_{n+1} \leftarrow c_n + a_n F_X(x_n)$

Interval width

n-th symbol to be encoded

Interval lower limit

Cumulative distribution (excluding current symbol)

$$F_X(\alpha_i) = \sum_{j=0}^{i-1} f_X(\alpha_j)$$

where $\mathcal{A}_X = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{K-1}\}$

Elias coding for changing probabilities

Elias coding algorithm

Start with unit interval

Initialize $c_0 = 0$ and $a_0 = 1$.

For each $n = 0, 1, \ldots$

Update $a_{n+1} \leftarrow a_n f_{X_n}(x_n)$

Update $c_{n+1} \leftarrow c_n + a_n F_{X_n}(x_n)$

Interval width

Interval lower limit

n-th symbol to be encoded

Cumulative distribution (excluding current symbol)

$$F_{X_n}(\alpha_i) = \sum_{j=0}^{i-1} f_{X_n}(\alpha_j)$$

where $\mathcal{A}_{X_n} = \{\alpha_0, \alpha_1, \alpha_2, ..., \alpha_{K-1}\}$

Elias coding for Markov random processes

Elias coding algorithm for Markov-p sources

Initialize $c_0 = 0$ and $a_0 = 1$.

For each $n = 0, 1, \ldots$

Set $p' = \min\{p, n\}$

Update $a_{n+1} \leftarrow a_n \, f_{X_n|\mathbf{X}_{n-p':n-1}}\left(x_n, \mathbf{x}_{n-p':n-1}\right)$

Update $c_{n+1} \leftarrow c_n + a_n F_{X_n|\mathbf{X}_{n-p':n-1}}\left(x_n, \mathbf{x}_{n-p':n-1}\right)$

Start with unit interval

Up to p previous symbols

Interval width

Interval lower limit

Cumulative conditional distribution (excluding current symbol)

n-th symbol to be encoded

Arithmetic coding

- Elias coding not practical for long symbol strings: required arithmetic precision grows with string length
- Finite precision implementations: *"arithmetic coding"*
- Widely used in modern image and video compression algorithms: JBIG, JPEG, JPEG-2000, H.263, H.264/AVC

Finite precision for arithmetic coding

N-bit precision is permissible, if rounding down

Initialize $c_0 = 0$ and $a_0 = 1$.

For each $n = 0, 1, \ldots$

Update $a_{n+1} \leftarrow a_n f_X(x_n)$

Update $c_{n+1} \leftarrow c_n + a_n F_X(x_n)$

At most $N+P$ LSBs change, except for a possible carry affecting MSBs.
Each bit can be affected by at most one carry.

P-bit approximation of probabilities

Finite precision for arithmetic coding (cont.)

- Output code string

$$c_n = .\underbrace{xxxxxxxxx\ldots xxx}_{\text{MSBs that will no longer change}}\underbrace{01111111\ldots1}_{r_n+1 \text{ bits that might be affected by a carry}}\underbrace{ccccccccc\ldots c}_{N+P \text{ LSBs that represent lower interval boundary for further computation}}$$

Send	Represent by state variable r_n	Represent by state variable C_n

- Maximum value of r_n can be limited by bit stuffing

Inefficiency due to interval rounding

- Recall: Subinterval of width $f_{\mathbf{X}}(\mathbf{x})$ is guaranteed to contain one number that can be represented by L_n binary digits, with

$$L_n \approx -\log_2 f_{\mathbf{X}_{0:n-1}}\left(\mathbf{x}_{0:n-1}\right)$$

- Hence, rounding one interval value a_{n+1} increases bit string by

$$\log_2 \frac{a_n f_X\left(x_n\right)}{a_{n+1}} < \log_2 \frac{2^{N-1}+1}{2^{N-1}} = \log_2\left(1+2^{1-N}\right) \approx \frac{1}{\ln 2} 2^{1-N}$$

Limited precision probabilities

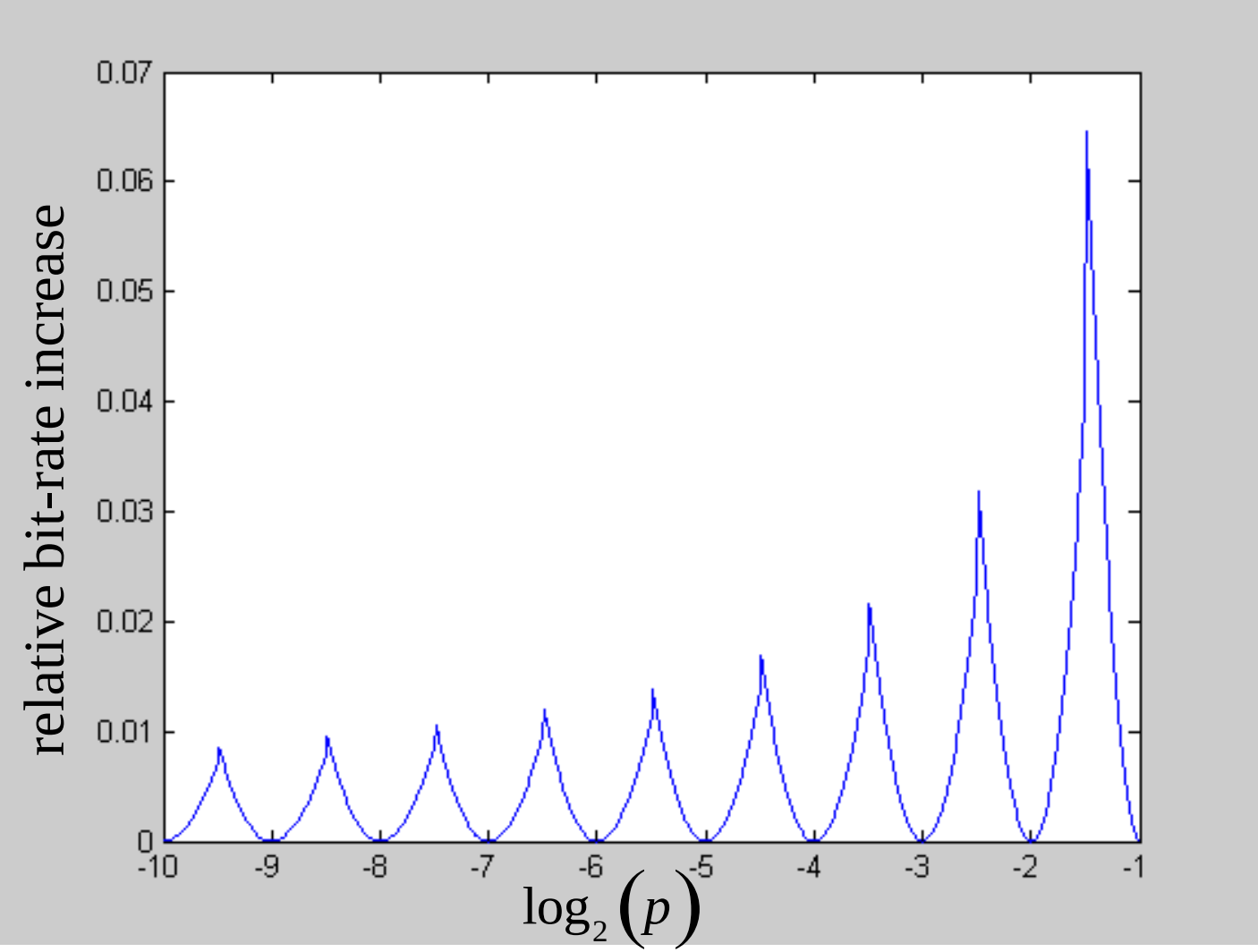
- Efficiency loss

$$E\left[-\log_2\left(f_X(x)\left(1+\frac{\Delta f_X(x)}{f_X(x)}\right)\right)\right] = \underbrace{E\left[-\log_2\left(f_X(x)\right)\right]}_{=H(X)} - \underbrace{E\left[\log_2\left(1+\frac{\Delta f_X(x)}{f_X(x)}\right)\right]}_{\approx \frac{1}{\ln 2} E\left[\frac{\Delta f_X(x)}{f_X(x)}\right]}$$

$$E\left[\frac{\Delta f_X(x)}{f_X(x)}\right] = \sum_x f_X(x)\cdot\frac{\Delta f_X(x)}{f_X(x)} = \sum_x \Delta f_X(x) = 0$$

provided that rounded probabilities still add to 1

Example: inefficiency for binary source due to representing smaller probability by 2^{-P}



Multiplication-free binary arithmetic coding

- Interval width is normalized to $1/2 \le A < 1$ by binary shift after each symbol
- Approximation, with $p(L) \le p(M)$:

$$\begin{aligned} Ap(L) &\approx \alpha \cdot p(L) = p'(L) \\ Ap(M) &= A(1-p(L)) \approx A - \alpha \cdot p(L) = A - p'(L) \end{aligned}$$

- Corresponds to approximating of $p(L)$ within a factor of 2:

$$p(L) \approx \frac{\alpha}{A} p(L)$$

- Factor α can be lumped into external probabilities
- Optimum α depends somewhat on data, typically

$$\alpha = \frac{2}{3} \quad \text{or} \quad \alpha = 0.708 \quad \text{or} \quad \alpha = \frac{3}{4}$$

JBIG

MPS-LPS switch

- Problem: interval width for symbol M can become negative

$$A - p'(L) < 0 \quad \text{if} \quad p'(L) > A \in \left[\frac{1}{2}, 1\right)$$

- Solution: switch the role of M and L, if $p'(L) > 1/2$

- Problem: interval for symbol M can be smaller than interval for L if

$$p'(L) > A - p'(L) \quad \Leftrightarrow \quad p'(L) > \frac{1}{2} A \in \left[1/4, 1/2\right)$$

- Solution: "Conditional exchange" of symbols, when problem occurs
- Improves compression efficiency, implemented in JPEG and JBIG

Sufficiency of binary coders

- Consider r.v. $X \in \{0,1,\ldots,2^K-1\}$ equivalent to a r.v. $\mathbf{B} = \begin{pmatrix} B_0 \\ B_1 \\ \vdots \\ B_{K-1} \end{pmatrix}$ (MSB … LSB)

- No loss by conditionally encoding bits (in any order)

$$H(X) = H(\mathbf{B})$$
$$= H(B_0) + H(B_1 \mid B_0) + \ldots + H(B_{K-1} \mid B_0, B_1, \ldots B_{K-2})$$

- Supply arithmetic coder with symbol/probability pairs

$$(b_0, p_0), (b_1, p_1), \ldots, (b_{K-1}, p_{K-1}) \quad \text{with } p_k = f_{B_k \mid \mathbf{B}_{0:k-1}}(0, \mathbf{b}_{0:k-1})$$

- Total number of (conditional) probabilities to be estimated per symbol x

$$1 + 2 + \cdots + 2^{K-1} = 2^K - 1$$

. . . same as with K-ary encoding of x

Adaptive probability estimation

- Consider stationary binary Markov-k process $\{X_n\}$ with probabilities

$$f_{X_k|\mathbf{X}_{0:k-1}}\left(0,\mathbf{x}_{0:k-1}\right)=f_{X_n|\mathbf{X}_{(n-k):n-1}}\left(0,\mathbf{x}_{(n-k):n-1}\right)$$

"Context vector," 2^k possible combinations

- Context labeling function, enumerating all possible context vectors

$$\lambda(\mathbf{x}):\ \mathbf{x}\in\{0,1\}^k\rightarrow\{0,1,\ldots,2^k-1\}$$

- Backward probability estimation separately for each context

Probability of symbol $x[n]=0$

$$p_0[n]=\frac{C_0\left[n,\lambda\left(\mathbf{x}_{(n-k):n-1}\right)\right]+\Delta}{\left(C_0\left[n,\lambda\left(\mathbf{x}_{(n-k):n-1}\right)\right]+\Delta\right)+\left(C_1\left[n,\lambda\left(\mathbf{x}_{(n-k):n-1}\right)\right]+\Delta\right)}$$

Count of zeroes that have previously occured in that context

Bias for low counts, typically $\Delta=1$

Adaptive probability estimation (cont.)

- Tracking changing statistics, compromise between
  - Rapid adaptation (small past sample counts)
  - Accurate estimates (large past sample counts)
- Scaled count estimation algorithm

Initialize $C_0 = C_1 = 0$

For $n = 0,1,\ldots$

If $x_n = 1$

$C_1 \leftarrow C_1 + 1$

else

$C_0 \leftarrow C_0 + 1$

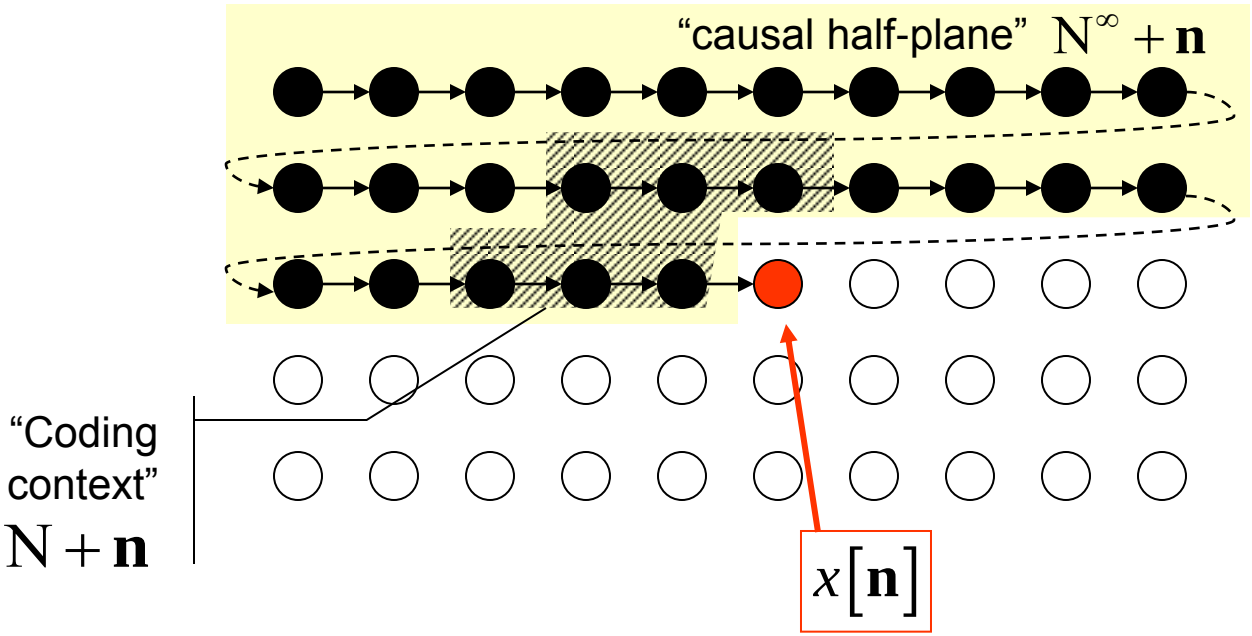
If $\min\{C_0, C_1\} > C_{\min}$ or $\max\{C_0, C_1\} > C_{\max}$

$$C_0 \leftarrow \left\lfloor \frac{C_0}{2} \right\rfloor; \quad C_1 \leftarrow \left\lfloor \frac{C_1}{2} \right\rfloor$$

$$\text{Estimate } p_0[n] = \frac{C_0 + \Delta}{C_0 + C_1 + 2\Delta}$$

Context adaptive coding

- 2-d extension of Markov model



- Markov conditional independence property

$$f_{X[\mathbf{n}]|\mathbf{X}_{\mathrm{N}^{\infty}+\mathbf{n}}}\left(X[\mathbf{n}],\mathbf{X}_{\mathrm{N}^{\infty}+\mathbf{n}}\right)=f_{X[\mathbf{n}]|\mathbf{X}_{\mathrm{N}+\mathbf{n}}}\left(X[\mathbf{n}],\mathbf{X}_{\mathrm{N}+\mathbf{n}}\right)$$

Context adaptive coding (cont.)

- Coding context vector can be mapped into context label directly without loss
  - Feasible for binary images
  - Example: JBIG uses 10 binary pixels as context label
- For 8-bit images, number of different contexts $256^{\|N\|}$
  - Context might have to be clustered
  - Combine with prediction

Reading

- W. B. Pennebaker, J. L. Mitchell, G. G. Langdon, Jr., R. B. Arps, “An overview of the basic principles of the Q-Coder adaptive binary arithmetic coder,” IBM J. Res. Develop., vol. 32, no. 6, November 1988.
- Witten, Radford, Neal, Cleary, “Arithmetic Coding for Data Compression” Communications of the ACM, vol. 30, no. 6, pp. 520-540, June 1987.
- Moffat, Neal, Witten, “Arithmetic Coding Revisited,” ACM Transactions on Information Systems, vol. 16, vo. 3, pp. 256–294, July 1998.