Homework #1 (34 pts)

Solutions

1. AWGN’s (9 pts)

Let $e$ be the Mean Square Error (MSE) of the linear Minimum MSE estimator of $x$ given $y$, i.e., $e = x - Wy$ where $W = R_{xy} R_{yy}^{-1}$. For jointly Gaussian $x$ and $y$, MSE $e$ is independent of the observation $y$, Hence $I(x; y)$ can be written as,

\[ I(x; y) = h(y) - h(y|x) \]
\[ = \log_2 (\pi e)^N |R_{yy}| - \log_2 (\pi e)^N |R_{nn}| \]
\[ = \log_2 \frac{|R_{yy}|}{|R_{nn}|} \]

\[ I(x; y) = h(x) - h(x - Wy|y) \]
\[ = \log_2 (\pi e)^N |R_{xx}| - \log_2 (\pi e)^N |R_{ee}| \]
\[ = \log_2 \frac{|R_{xx}|}{|R_{ee}|} \]

where $R_{yy}, R_{nn}, R_{xx}$ and $R_{ee} \in \mathbb{C}^{N \times N}$ are covariance matrices of $y, n, x$ and $e$ respectively. Therefore,

\[ I(x; y) = \log_2 \frac{|R_{yy}|}{|R_{nn}|} = \log_2 \frac{|R_{xx}|}{|R_{ee}|} \]

For a real baseband channel,

\[ I(x; y) = \frac{1}{2} \log_2 \frac{|R_{yy}|}{|R_{nn}|} = \frac{1}{2} \log_2 \frac{|R_{xx}|}{|R_{ee}|} \]

(From Chapter 5.)

2. multiple-user channel types - row vector (7 pts)

(a) Multiple Access Channel (MAC)

(b) $y = Hx + n = x_1 + 2x_2 + n$, where $H = [1 \ 2], x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Then,

\[ I(x; y) = \frac{1}{2} \log_2 \frac{\sigma_{yy}^2}{\sigma_{nn}^2} = \frac{1}{2} \log_2 \frac{\mathcal{E}_{y,1} + 4 \cdot \mathcal{E}_{y,2} + \sigma^2}{\sigma^2} = \frac{1}{2} \log_2 \frac{1 + 4 + .001}{.001} = 6.14. \]
(c) Yes, decode either user first and subtract his components from the received signal and then decode the other user.

3. *multiple-user channel types - column vector (8 pts)*

(a) Broadcast Channel (BC)

(b) \( y = Hx + n = \begin{bmatrix} 1 \\ 2 \end{bmatrix} x + n \)

Then,

\[
I(x; y) = \frac{1}{2} \log_2 \frac{|R_{yy}|}{|R_{nn}|} = \frac{1}{2} \log_2 \frac{|H^*E_xH + \sigma^2 I_2|}{|\sigma^2 I_2|} \\
= \frac{1}{2} \log_2 \left[ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot 2 \cdot [1 \ 2] + .001 \cdot I_2 \right] \\
= 6.64
\]

(c) No, in general this mutual information is not achievable since coordination is not possible at the receivers for a broadcast channel. However with some conditions on receivers’ noise correlation coefficient, this mutual information can be achievable. You will see more on this in homework set 3.

4. (a) BC, 2 users.

(b) MAC, 2 users.

(c) It can be formed as a 3-user IC problem.

(d) The recursive representation can be formed as follows:

\[
\begin{bmatrix} B_s \\ U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & 0 & 0 \\ 0 & 0 & H_{33} & 0 \\ 0 & 0 & 0 & H_{44} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ B_s \\ B_s \end{bmatrix} = \begin{bmatrix} H_{11}^* & 0 & 0 \\ 0 & H_{21}^* & 0 \\ 0 & 0 & H_{33}^* & 0 \\ 0 & 0 & 0 & H_{44}^* \end{bmatrix} \begin{bmatrix} B_s \\ U_1 \\ U_2 \end{bmatrix}
\]