PE 281 - APPLIED MATHEMATICS IN RESERVOIR ENGINEERING

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These notes were to developed accompany the lecture material for PE281. I've tried hard to avoid losing negative signs etc. but some typos may have snuck through. If you do find errors please contact me.

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Chapter 1

Introduction

1.1 The Diffusion Equation

This course considers slightly compressible fluid flow in porous media. The differential equation governing the flow can be derived by performing a mass balance on the fluid within in a control volume.

1.1.1 One-dimensional Case

First consider a one-dimensional case as shown in Figure 1.1:

$$(mass\ \ in) - (mass\ \ out) = (mass\ \ accumulation) \tag{1.1}$$

$$\Rightarrow \Delta t q\rho|_x - \Delta t q\rho|_{x+\Delta x} = \phi V\rho|_{t+\Delta t} - \phi V\rho|_t \tag{1.2}$$

where $V = \Delta x A$ and $q = -\frac{kA}{\mu}\frac{\partial p}{\partial x}$
Dividing (1.2) through by Δx and Δt and taking limits as $\Delta x \rightarrow 0$ and $\Delta t \rightarrow 0$ gives:

$$\lim_{\Delta x \rightarrow 0} \frac{q\rho|_x - q\rho|_{x+\Delta x}}{\Delta x} = \lim_{\Delta t \rightarrow o} \frac{\phi A\rho|_{t+\Delta t} - \phi A\rho|_t}{\Delta t} \tag{1.3}$$

$$\Rightarrow -\frac{\partial}{\partial x}(q\rho) = \frac{\partial}{\partial t}(\phi A\rho) \tag{1.4}$$

Substituting Darcy's law into (1.4) gives:

$$\frac{\partial}{\partial x}\left(\frac{kA}{\mu}\rho\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial t}(\phi A\rho) \tag{1.5}$$

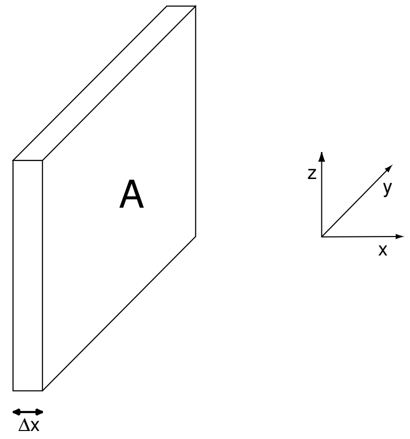

Figure 1.1: One-dimensional control volume

Now assume (for simplicity) that k, μ and A are constant:

$$\Rightarrow \frac{\partial}{\partial x}\left(\rho\frac{\partial p}{\partial x}\right) = \frac{\mu}{k}\frac{\partial \phi\rho}{\partial t} \tag{1.6}$$

Now account for the dependence of ρ on pressure by introducing the isothermal compressibility:

$$c = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T \tag{1.7}$$

where T denotes that the derivative is taken at constant temperature. Equation (1.7) defines and EOS (equation of state):

$$\int_{p_{sc}}^{p} cdP = \int_{\rho_{sc}}^{\rho}\frac{d\rho}{\rho} \tag{1.8}$$

$$\Rightarrow c(p - p_{sc}) = ln\rho - ln\rho_{sc} \tag{1.9}$$

$$\Rightarrow \rho = \rho_{sc}e^{c(p-p_{sc})} \tag{1.10}$$

Now substitute $\rho(p)$ (equation 1.10) into equation 1.6):

$$\frac{\partial}{\partial x}\left(\rho_{sc}e^{c(p-p_{sc})}\frac{\partial p}{\partial x}\right) = \frac{\mu}{k}\left(\rho_{sc}e^{c(p-p_{sc})}\frac{\partial \phi}{\partial t} + \phi\frac{\partial}{\partial t}\left(\rho_{sc}e^{c(p-p_{sc})}\right)\right) \tag{1.11}$$

The right hand side terms in equation 1.11 require further attention. First consider the final term, $\phi\frac{\partial \rho}{\partial t}$:

$$\frac{\partial \rho}{\partial t} = \rho_{sc}e^{c(p-psc)}c\frac{\partial p}{\partial t} \tag{1.12}$$

Now consider $\frac{\partial \phi}{\partial t}$. First define the rock compressibility as:

$$c_r = \frac{1}{\phi}\left(\frac{\partial \phi}{\partial p}\right)_T \tag{1.13}$$

$$\Rightarrow \frac{\partial \phi}{\partial t} = \phi c_r \frac{\partial p}{\partial t} \tag{1.14}$$

Substitute equation 1.12 and 1.14 into 1.11:

$$\frac{\partial}{\partial x}\left(\rho \frac{\partial p}{\partial x}\right) = \frac{\mu}{k}\rho\phi(c_r + c)\frac{\partial p}{\partial t} \tag{1.15}$$

Let $c_t = c_r + c$. Now expand the spatial derivative in equation 1.15:

$$\rho \frac{\partial^2 p}{\partial x^2} + \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial x} = \frac{\phi \mu c_t}{k}\rho \frac{\partial p}{\partial t} \tag{1.16}$$

Now consider the second term in equation 1.16:

$$\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial x} = \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial p}\frac{\partial p}{\partial x} = \frac{\partial \rho}{\partial p}\left(\frac{\partial p}{\partial x}\right)^2 \tag{1.17}$$

This term is expected to be small so it is usually neglected.
Finally we have:

$$\frac{\partial^2 p}{\partial x^2} = \frac{\phi \mu c_t}{k}\frac{\partial p}{\partial t} \tag{1.18}$$

1.2 Three-dimensional Case

The diffusion equation can be expressed using the notation of vector calculus for a general coordinate system as:

$$\nabla^2 p = \frac{\phi \mu c_t}{k}\frac{\partial p}{\partial t} \tag{1.19}$$

For the case of the radial coordinates the diffusion equation is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2} = \frac{\phi \mu c_t}{k}\frac{\partial p}{\partial t} \tag{1.20}$$

1.3 Dimensionless Form

1.3.1 One Dimensional Problem

The pressure equation for one dimensional flow (equation 1.18) can be written in dimensionless form by choosing the following dimensionless variables:

$$p_D = \frac{p_i - p}{p_i} \tag{1.21}$$

$$x_D = \frac{x}{L} \tag{1.22}$$

where L is a length scale in the problem.

$$t_D = \frac{kt}{\phi\mu c_t L^2} \tag{1.23}$$

With this choice of dimensionless variables the flow equation becomes:

$$\frac{\partial^2 p_D}{\partial x_D^2} = \frac{\partial p_D}{\partial t_D} \tag{1.24}$$

1.3.2 Radial Problem

The radial form of the pressure equation is usually written in nondimensional form taking account of the boundary conditions. When only radial variations of pressure are considered the pressure equation is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = \frac{\phi\mu c_t}{k}\frac{\partial p}{\partial t} \tag{1.25}$$

Boundary and initial conditions:

$$q = -\frac{2\pi kh}{\mu} r\frac{\partial p}{\partial r}, \qquad r = r_w \tag{1.26}$$

$$p = p_i, \qquad r \to \infty, \forall t \tag{1.27}$$

$$p = p_i, \qquad t = 0, \forall r \tag{1.28}$$

Begin by setting:

$$p_D = \alpha(p_i - p) \tag{1.29}$$

where alpha must still be determined. The infinite acting boundary condition becomes:

$$p_D = \alpha(p_i - p_i) = 0, \qquad r \to \infty, \forall t \tag{1.30}$$

The initial condition becomes:

$$p_D = \alpha(p_i - p_i) = 0, \qquad t = 0, \forall r \tag{1.31}$$

Set the dimensionless length, r_D as:

$$r_D = \frac{r}{r_w} \tag{1.32}$$

Set dimensionless time, t_D as:

$$t_D = \frac{t}{t^*} \tag{1.33}$$

where t^* must still be determined. Now substitute p_D and r_D into the pressure equation:

$$\frac{1}{r_D r_w}\frac{\partial}{\partial r_D r_w}\left(r_D r_w \frac{\partial}{\partial r_D r_w}\left(-\frac{p_D}{\alpha} + p_i\right)\right) = \frac{\phi\mu c_t}{k}\frac{\partial\left(-\frac{p_D}{\alpha} + p_i\right)}{\partial t_D t^*} \tag{1.34}$$

Simplifying (1.34) gives:

$$\frac{1}{r_w^2\alpha}\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial p_D}{\partial r_D}\right) = \frac{\phi\mu c_t}{kt^*\alpha}\frac{\partial p_D}{\partial t_D} \tag{1.35}$$

$$\Rightarrow t^* = \frac{\phi\mu c_t r_w^2}{k} \tag{1.36}$$

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial p_D}{\partial r_D}\right) = \frac{\partial p_D}{\partial t_D} \tag{1.37}$$

Finally determine α from the inner boundary condition:

$$q = \frac{2\pi kh}{\mu} r\frac{\partial p}{\partial r} \tag{1.38}$$

(No negative sign is required here because the flow is in the negative r direction)

$$\frac{q\mu}{2\pi kh} = r_D r_w \frac{\partial}{\partial r_D r_w}\left(-\frac{p_D}{\alpha} + p_i\right) \tag{1.39}$$

$$\Rightarrow \alpha = \frac{2\pi kh}{q\mu} \tag{1.40}$$

Nondimensionalise inner boundary condition:

$$\frac{\partial p_D}{\partial r_D}|_{r_D=1} = -1 \tag{1.41}$$

1.4 Superposition

Solutions to complex problems can be found by adding simple solutions representing the pressure distribution due to wells producing at constant rate at various locations and times. This concept is known as superposition. It is only applicable to linear problems.

Superposition in Time

Assume we have an analytical solution, $p^{const}(q, r, t)$, to the problem of a well producing at a constant rate in a given reservoir. Using superposition in time this solution can be extended to handle a well with a variable flow rate. If a well begins producing at rate q_1 then at time t_1 the rate changes to q_2 the flow rate can be represented as shown in Figure 1.2.

The analytical solution for the pressure distribution caused by the well producing at variable rate is:

$$p^{var}(r, t) = p^{const}(q_1, r, t) + p^{const}(q_2 - q_1, r, t - t_1) \tag{1.42}$$

Superposition in Space

Production from multiple wells can be handled using superposition also. Suppose again we have a solution $p^{const}(q, r, t)$ for the pressure distribution due to a well located at the origin, flowing at rate q. The solution for a a reservoir containing two wells as shown in Figure 1.3 can be generated by summing this solution as follows:

$$p(r, t) = p^{const}(q1, r1, t) + p^{const}(q2, r2, t) \tag{1.43}$$

where

$$r1 = \sqrt{(x - x1)^2 + (y - y1)^2} \tag{1.44}$$

$$r2 = \sqrt{(x - x2)^2 + (y - y2)^2} \tag{1.45}$$

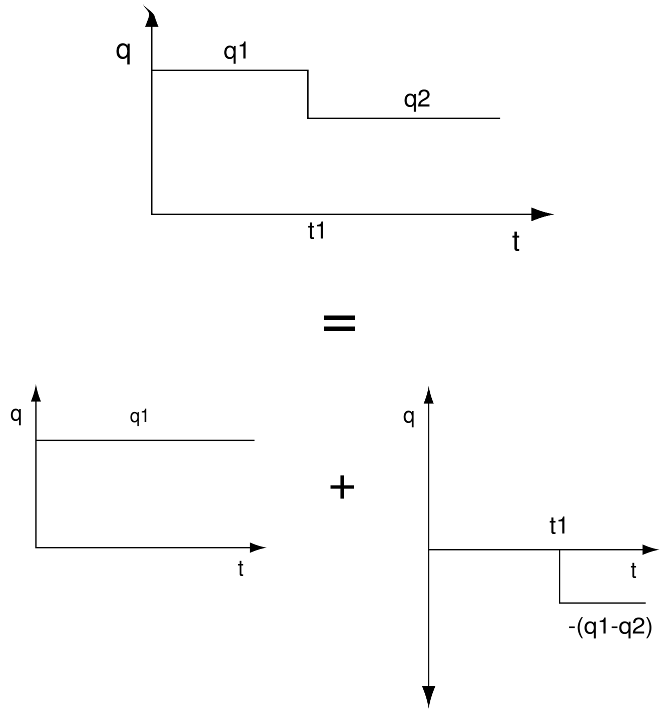

Figure 1.2: Flow rate variation

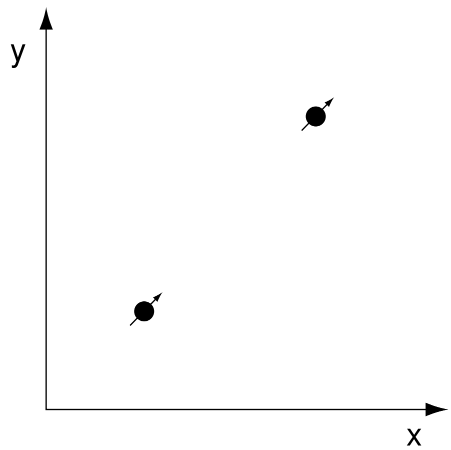

Figure 1.3: Well configuration

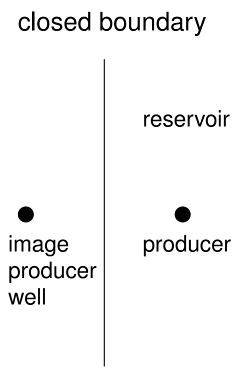

Figure 1.4: Closed boundary

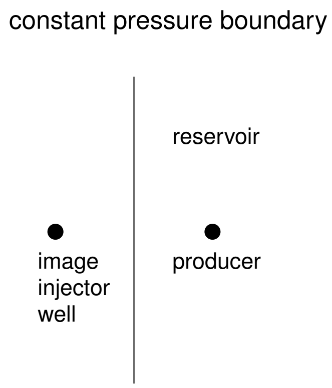

Figure 1.5: Constant pressure boundary

Using Superposition to Handle Boundary Conditions

Superposition in space can be used to impose constant pressure and/or closed boundary conditions. To do so fictitious wells known as image wells are placed in the reservoir in such a way that their effect on the pressure distribution creates the boundary condition. If multiple boundary conditions are involved this can lead to an array of images wells whose contribution to the reservoir pressure distribution is summed.

Examples of the use of image wells are shown in Figures 1.4 and 1.5.

1.4.1 Well Boundary Conditions when Superposition is Applied

The superposition theorem guarantees the pressure distribution obtaining by summing simple solutions will satisfy the pressure equation. The boundary condition at the well however requires careful consideration.

Wells Controlled by Bottom Hole Pressure

If a reservoir contains two wells with specified bottom hole pressures p_1 and p_2 a pressure solution can be obtained by summing two solutions for a single well at specified well pressure. This solution will satisfy the pressure equation. However this solution **will not** satisfy the required bottom hole pressures at the wells. If both wells are producers then there will be additional drawdowns at each well due to production in the other well. However if the wells are far apart this effect is likely to be small.

Wells with a Specified Flow Rate

A solution for a reservoir with multiple wells with specified flow rates can be generated from solutions for a single well. Unlike the case of bottom hole pressure controlled wells this solution **will** satisfy the flow rate boundary condition at each well. This is possible because each superposed solution conserves mass locally so it does not add any extra flow at the well locations.

Chapter 2

The Laplace Transform

The Laplace Transform is defined by:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \hat{f}(s) \tag{2.1}$$

Example:

$$f(t) = t \tag{2.2}$$

Use integration by parts, recall:

$$\int_a^b u \frac{dv}{dt} dt = uv|_a^b - \int_a^b v \frac{du}{dt} dt \tag{2.3}$$

Choose $u = t$ and $v = -\frac{1}{s} e^{-st}$

$$\Rightarrow \mathcal{L}\{f(t)\} = -\frac{t}{s} e^{-st}|_0^\infty + \int_0^\infty \frac{1}{s} e^{-st} dt \tag{2.4}$$

$$= 0 + \left[-\frac{1}{s^2} e^{-st}\right]_0^\infty = \frac{1}{s^2} \tag{2.5}$$

For the Laplace transform to exist the following requirements must hold:
a) $f(t)$ have a finite number of maxima, minima and discontinuities
b) there exist constants α, M, T such that

$$e^{-\alpha t} |f(t)| < M, \qquad t > T \tag{2.6}$$

Functions satisfying this requirement are known as functions of exponential order. For $t > 0$ there is $\alpha_1 > \alpha$ such that:

$$e^{-\alpha_1 t} |f(t)| < M \tag{2.7}$$

When this just holds α is known as the abscissa of covergence.
Example:

$$f(t) = e^{2t} \tag{2.8}$$

$$e^{-\alpha t}e^{2t} = e^{-(\alpha-2)t} \tag{2.9}$$

(2.9) remains bounded for $\alpha \leq 2$, therefore the abscissa of convergence for this $f(t)$ is 2.

2.1 Properties of Laplace Transforms

2.1.1 Theorem 1 - Linearity of the Laplace Transform Operator

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\} \tag{2.10}$$

$\Rightarrow$ the Laplace Transform is a linear operator.
Proof:

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = \int_0^\infty e^{-st} c_1 f_1 + c_2 f_2 dt \tag{2.11}$$

$$= c_1 \int_0^\infty e^{-st} f_1 dt + c_2 \int_0^\infty e^{-st} f_2 dt \tag{2.12}$$

$$= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\} \tag{2.13}$$

2.1.2 Theorem 2 - Laplace Transform of a Time Derivative

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \tag{2.14}$$

Proof:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t) dt \tag{2.15}$$

Integrate by parts

$$= e^{-st} f(t)|_0^\infty - \int_0^\infty f(t)(-se^{-st}) dt \tag{2.16}$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t) dt \tag{2.17}$$

$$= s\mathcal{L}\{f(t)\} - f(0) \tag{2.18}$$

2.1.3 Theorem 3 - Laplace Transform of a Derivative

$$\mathcal{L}\left\{\frac{\partial^n f}{\partial t^n}\right\} = s^n\mathcal{L}\{f(t)\} - \sum_{i=0}^{n-1} s^i f^{n-i-1}(0) \tag{2.19}$$

This can be proved by repeated application of Theorem 2.

2.1.4 Theorem 4 - Early Time Behaviour

$$\lim_{s\to\infty} s\mathcal{L}\{f(t)\} = \lim_{t\to 0^+} f(t) = f(0^+) \tag{2.20}$$

Proof:
Begin from Theorem 2

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^+) \tag{2.21}$$

Now take limits:

$$\lim_{s\to\infty} \mathcal{L}\{f'(t)\} = \lim_{s\to\infty} s\mathcal{L}\{f(t)\} - f(0^+) \tag{2.22}$$

If $f'(t)$ is of exponential order:

$$\lim_{s\to\infty} e^{-st} f'(t)dt \to 0 \tag{2.23}$$

$\Rightarrow$

$$|f'(t)| < Me^{\alpha t}, \qquad \forall t > 0 \tag{2.24}$$

$$I(b) = \int_0^b |f'(t)|e^{-st}dt < \int_0^b Me^{\alpha t}e^{-st}dt = \left.\frac{Me^{-(s-\alpha)t}}{-(s-\alpha)}\right|_0^b \tag{2.25}$$

$s > \alpha$ as $b \to \infty$
$\Rightarrow$

$$\lim_{b\to\infty} I(b) = \frac{M}{s-\alpha} \tag{2.26}$$

then as $s \to \infty, I(b) \to 0$
Therefore the left hand side of (2.21) tends to zero, i.e.:

$$0 = \lim_{s\to\infty} s\mathcal{L}\{f(t)\} - f(0^+) \tag{2.27}$$

proving theorem 4.

2.1.5 Theorem 5 - Late Time Behaviour

$$\lim_{s\to 0} s\mathcal{L}\{f(t)\} = \lim_{t\to\infty} f(t) \tag{2.28}$$

Proof:
Begin from Theorem 2

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^+) \tag{2.29}$$

Now take limits:

$$\lim_{s\to 0} \mathcal{L}\{f'(t)\} = \lim_{s\to 0} s\mathcal{L}\{f(t)\} - f(0^+) \tag{2.30}$$

Expand the left hand side term:

$$\lim_{s\to 0} \int_0^\infty e^{-st} f'(t)dt = \int_0^\infty f'(t) lim_{s\to 0} e^{-st} dt = \int_0^\infty f'(t)dt \tag{2.31}$$

$$= lim_{t\to\infty} f(t) - f(0) \tag{2.32}$$

Substituting (2.32) into (2.29) gives:

$$\lim_{s\to 0} s\mathcal{L}\{f(t)\} = \lim_{t\to\infty} f(t) \tag{2.33}$$

2.1.6 Theorem 6 - Multiplication of a Transform by s

If $\mathcal{L}\{f(t)\} = s\phi(s)$ then $f(t) = \frac{\partial}{\partial t}\mathcal{L}^{-1}\{\phi(s)\}$
Proof:

$$F(t) = \mathcal{L}^{-1}\{\phi(s)\} \tag{2.34}$$

Use Theorem 2:

$$\mathcal{L}\{F'(t)\} = s\mathcal{L}\{F(t)\} - F(0) \tag{2.35}$$

and use Theorem 4:

$$F(0) = \lim_{s\to\infty} s\mathcal{L}\{F(t)\} = \lim_{s\to\infty} s\phi(s) \tag{2.36}$$

$$= \lim_{s\to\infty} \mathcal{L}\{f(t)\} = 0 \tag{2.37}$$

Now consider $F'(t)$ by returning to Theorem 2:

$$\mathcal{L}\{F'(t)\} = s\mathcal{L}\{F(t)\} = s\phi(s) = \mathcal{L}\{f(t)\} \tag{2.38}$$

Taking inverse Laplace Transforms of this gives:

$$\mathcal{L}^{-1}\{\mathcal{L}\{F'(t)\}\} = \mathcal{L}^{-1}\{s\phi(s)\} = \mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\} \tag{2.39}$$

$\Rightarrow$

$$F'(t) = \mathcal{L}^{-1}\{s\phi(s)\} = f(t) \tag{2.40}$$

$\Rightarrow$

$$\frac{\partial}{\partial t}\mathcal{L}\{\phi(s)\} = f(t) \tag{2.41}$$

Example: Suppose we want to find the inverse transform of:

$$\mathcal{L}\{f(t)\} = \frac{s}{s^2 + a^2} \tag{2.42}$$

We know can use the following transform relationship to help us:

$$\mathcal{L}\{\frac{\sin at}{a}\} = \frac{1}{s^2 + a^2} \tag{2.43}$$

Using theorem 6 we know:

$$f(t) = \frac{\partial}{\partial t}\frac{\sin at}{a} = \cos at \tag{2.44}$$

2.1.7 Theorem 7 - Division of a Transform by s

$$\mathcal{L}\left\{\int_0^t f(t)dt\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} \tag{2.45}$$

Proof:

$$\mathcal{L}\left\{\int_0^t f(t)dt\right\} = \int_0^\infty e^{-st}\left(\int_0^t f(t')dt'\right)dt \tag{2.46}$$

Use integration by parts:

$$= -\left[\frac{1}{s}\int_0^t f(t')dt'\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}f(t)dt = \frac{1}{s}\mathcal{L}\{f(t)\} \tag{2.47}$$

Example: Suppose we require the inverse transform of:

$$\mathcal{L}\{f(t)\} = \frac{1}{s^3 + 4s} = \frac{1}{s}\frac{1}{s^2 + 4} \tag{2.48}$$

We know:

$$\mathcal{L}\left\{\frac{1}{s^2 + 4}\right\} = \frac{1}{2}\sin 2t \tag{2.49}$$

$$\Rightarrow f(t) = \int_0^t \frac{1}{2}\sin 2t dt = \frac{1}{4}(1 - \cos 2t) \tag{2.50}$$

2.1.8 Theorem 8 - First Shift Theorem

$$\mathcal{L}\{e^{-at}f(t)\} = \hat{f}(s+a) \tag{2.51}$$

Proof:

$$\mathcal{L}\{e^{-at}f(t)\} = \int_0^\infty e^{-at}e^{-st}f(t)dt = \int_0^\infty e^{-(s+a)}f(t)dt = \hat{f}(s+a) \tag{2.52}$$

2.1.9 Theorem 9 - Second Shift Theorem

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\} \tag{2.53}$$

where $u(t-a)$ is a unit step function.

$$\begin{aligned} u(t-a) &= 1, \quad t-a>0 && (2.54)\\ &= 0, \quad otherwise && (2.55)\\ & && (2.56) \end{aligned}$$

Proof:

$$\begin{aligned} \mathcal{L}\{f(t-a)u(t-a)\} &= \int_0^\infty f(t-a)u(t-a)e^{-st}dt && (2.57)\\ &= \int_a^\infty f(t-a)e^{-st}dt && (2.58)\\ &= \int_0^\infty f(\tau)e^{-s(\tau+a)}d\tau && (2.59) \end{aligned}$$

where $\tau = t-a$ Apply Theorem 8:

$$\begin{aligned} &= e^{-sa}\int_0^\infty f(\tau)e^{-s\tau}d\tau && (2.60)\\ &= e^{-sa}Lf(t)dt && (2.61) \end{aligned}$$

2.1.10 Theorem 10 - Multiplication by t

$$\mathcal{L}\{tf(t)\} = -\hat{f}'(s) \tag{2.62}$$

Proof:

$$\begin{aligned} \hat{f}'(s) &= \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt && (2.63)\\ &= \int_0^\infty -te^{-st}f(t)dt && (2.64)\\ &= -\mathcal{L}\{tf(t)\} && (2.65) \end{aligned}$$

2.1.11 Theorem 11 - Division by t

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \hat{f}(s)ds \tag{2.66}$$

Proof:

$$\int_s^\infty \hat{f}(s)ds = \int_s^\infty \int_0^\infty e^{-st} f(t)dtds \tag{2.67}$$

$$= \int_0^\infty \int_s^\infty e^{-st} f(t)dsdt \tag{2.68}$$

$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt \tag{2.69}$$

$$= \int_0^\infty \frac{f(t)}{t} e^{-st}dt \tag{2.70}$$

$$= \mathcal{L}\left\{\frac{f(t)}{t}\right\} \tag{2.71}$$

2.1.12 Theorem 12 - Convolution

$$\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \mathcal{L}\{\int_0^t f(t-\lambda)g(\lambda)d\lambda\} \tag{2.72}$$

$$= \mathcal{L}\{\int_0^t f(\lambda)g(t-\lambda)d\lambda\} \tag{2.73}$$

$$= \mathcal{L}\{f(t) * g(t)\} \tag{2.74}$$

Proof:
First use the definition of the Laplace transform:

$$\mathcal{L}\{\int_0^t f(t-\lambda)g(\lambda)d\lambda\} = \int_0^\infty \int_0^t f(t-\lambda)g(\lambda)e^{-st}d\lambda dt \tag{2.75}$$

Change limits on the λ integral by introducing a step function:

$$= \int_0^\infty \int_0^\infty u(t-\lambda)f(t-\lambda)g(\lambda)e^{-st}d\lambda dt \tag{2.76}$$

(See Figure 2.1 for the step function.)
Change the order of integration:

$$= \int_0^\infty g(\lambda)\int_0^\infty u(t-\lambda)f(t-\lambda)e^{-st}dtd\lambda \tag{2.77}$$

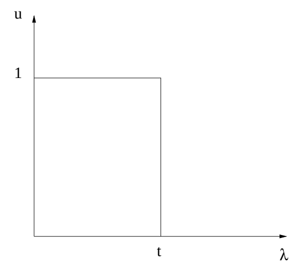

Figure 2.1: Step function as a function of λ

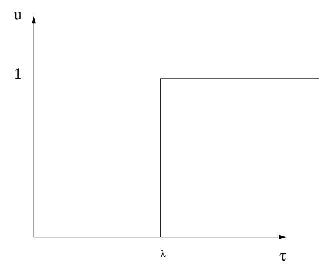

Figure 2.2: Step function as a function of t

(See Figure 2.2 for the step function.)
Take account of step function:

$$= \int_0^\infty g(\lambda) \int_\lambda^\infty f(t-\lambda)e^{-st}dtd\lambda \tag{2.78}$$

Apply first shift theorem:

$$= \int_0^\infty g(\lambda) \left[\int_0^\infty f(\tau)e^{-s(\tau+\lambda)}d\tau\right] d\lambda \tag{2.79}$$

$$= \int_0^\infty g(\lambda)e^{-s\lambda}d\lambda \int_0^\infty f(\tau)e^{-s\tau}d\tau \tag{2.80}$$

$$= \mathcal{L}\{g(t)\}\mathcal{L}\{f(t)\} \tag{2.81}$$

where $\tau = t - \lambda$

2.2 Solving Differential Equations with Laplace Transforms

Laplace transforms can be used as a powerful tool to solve differential equations. The general procedure is:
- transform both side of the equation
- solve the transformed equation to get an expression for the Laplace transform of the solution
- invert to find the solution in real space

This approach turns an ordinary differential equation into an alegbraic equation and a partial differential equation in x and t into an ordinary differential equation in x or t.

2.2.1 Ordinary Differential Equation Example

Solve:

$$y'' + 2y' + y = te^{-t} \tag{2.82}$$

where

$$y(t = 0) = 1 \tag{2.83}$$

$$y'(t = 0) = -2 \tag{2.84}$$

$$\mathcal{L}\{y''\} = s^2\hat{y} - sy(0) - y'(0) \tag{2.85}$$

$$\mathcal{L}\{y'\} = s\hat{y} - y(0) \tag{2.86}$$

$$\mathcal{L}\{te^{-t}\} = \frac{1}{(s+1)^2} \tag{2.87}$$

$$\Rightarrow s^2\hat{y} - s + 2 + 2s\hat{y} - 2 + \hat{y} = \frac{1}{(s+1)^2} \tag{2.88}$$

Solve for $\hat{y}$:

$$(s^2 + 2s + 1)\hat{y} - s = \frac{1}{(s+1)^2} \tag{2.89}$$

$$(s+1)\hat{y} = \frac{1}{(s+1)^2} + s \tag{2.90}$$

$$\Rightarrow \hat{y} = \frac{1}{(s+1)^4} + \frac{s}{(s+1)^2} \tag{2.91}$$

Now invert to find y. Consider the first term, we know (from tables):

$$\mathcal{L}\{t^(n)\} = \frac{n!}{s^{n+1}} \tag{2.92}$$

Combining this transform with the first shift theorem gives:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^4}\right\} = \frac{e^{-t}t^3}{3!} \tag{2.93}$$

Now consider the second term:

$$\frac{s}{(s+1)^2} = \frac{s+1-1}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2} \tag{2.94}$$

We can use the known transforms:

$$\mathcal{L}\{1\} = \frac{1}{s} \tag{2.95}$$

and

$$\mathcal{L}\{t\} = \frac{1}{s^2} \tag{2.96}$$

Combining this with the first shift theorem again gives:

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\} = e^{-t} - e^{-t}t \tag{2.97}$$

The final solution for y is:

$$y = e^{-t}\left(\frac{t^3}{3!} - t + 1\right) \tag{2.98}$$

2.3 Computing Laplace Transforms in Mathematica

Laplace transforms can be computed using Mathematica using the Laplace Transform Calculus package. On wasson the example given in Equation (2.5) can be computed using:

```
In[1]:= Needs["Calculus`LaplaceTransform`"]

In[2]:= LaplaceTransform[t,t,s]

          -2
Out[2]= s
```

The inverse transform can be computed using:

```
In[4]:= InverseLaplaceTransform[s^-2,s,t]

Out[4]= t
```

If you're using the Mathematica version available on WinDD there is no need to load the Calculus`LaplaceTransform` package.
Note: For problem sets please work out any transforms required by hand and show working, unless otherwise specified. Feel free to check your work against Mathematica output however.

Chapter 3

Petroleum Engineering Applications of Laplace Transforms

This chapter outlines how Laplace transforms can be used to solve problems of interest to petroleum engineers. The solutions presented consider different treatments of the well and different boundary conditions.

3.1 Line Source Solution

This section considers infinite acting radial flow in a reservoir where the well is modelled as a line source. The differential equation and boundary conditions involved are:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = \frac{\phi\mu c_t}{k}\frac{\partial p}{\partial t} \tag{3.1}$$

A constant rate boundary condition is specified at $r = 0$.

$$q = \frac{2\pi kh}{\mu} r\frac{\partial p}{\partial r} \tag{3.2}$$

The outer boundary condition is:

$$p = p_i, \qquad r \to \infty \tag{3.3}$$

The initial condition is:

$$p = p_i, \qquad \forall r, t = 0 \tag{3.4}$$

This can be written in dimensionless form as:

$$p_D = \frac{2\pi kh}{q\mu}(p_i - p) \tag{3.5}$$

$$r_D = \frac{r}{r_w} \tag{3.6}$$

$$t_D = \frac{kt}{\phi\mu c_t r_w^2} \tag{3.7}$$

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial p_D}{\partial r_D}\right) = \frac{\partial p_D}{\partial t_D} \tag{3.8}$$

$$p_D(r_D, t_D = 0) = 0 \tag{3.9}$$

$$p_D(r_D \to \infty, t_D) = 0 \tag{3.10}$$

$$r_D\frac{\partial p_D}{\partial t_D}|_{r_D \to 0} = -1 \tag{3.11}$$

The solution procedure begins by Laplace transforming both sides of the pressure equation:

$$\mathcal{L}\left\{\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial p_D}{\partial r_D}\right)\right\} = \mathcal{L}\left\{\frac{\partial p_D}{\partial t_D}\right\} \tag{3.12}$$

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial \hat{p}_D}{\partial r_D}\right) = s\hat{p}_D - p_D(r_D, t_D = 0) \tag{3.13}$$

$$\Rightarrow \frac{\partial^2 \hat{p}_D}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial \hat{p}_D}{\partial r_D} - s\hat{p}_D = 0 \tag{3.14}$$

A solution to this differential equation can be found by noting that it is an example of a modified Bessel equation.

3.1.1 Bessel and Modified Bessel Equations

The Bessel equation is:

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \tag{3.15}$$

$$y = c_1 J_n(x) + c_2 Y_n(x) \tag{3.16}$$

where J_n is a Bessel function of the first kind of order n and Y_n is a Bessel function of the second kind or order n.
The modified Bessel equation is:

$$x^2y'' + xy' - (x^2 + n^2)y = 0 \tag{3.17}$$

$$y = c_1 I_n(x) + c_2 K_n(x) \tag{3.18}$$

where I_n and K_n are modified Bessel functions of order n.

3.1.2 Laplace Space Solution for p_D

The transformed pressure equation can be written as:

$$r_D^2 \frac{\partial^2 \hat{p}_D}{\partial r_D^2} + r_D \frac{\partial \hat{p}_D}{\partial r_D} - r_D^2 s \hat{p}_D = 0 \tag{3.19}$$

Substitute $\xi = r_D\sqrt{s}$:

$$\xi^2 \frac{\partial^2 \hat{p}_D}{\partial \xi^2} + \xi \frac{\partial \hat{p}_D}{\partial \xi} - \xi^2 \hat{p}_D = 0 \tag{3.20}$$

Solve for $\hat{p}_D$:

$$\hat{p}_D = c_1 I_0(r_D\sqrt{s}) + c_2 K_0(r_D\sqrt{s}) \tag{3.21}$$

Now consider the boundary conditions. First consider the infinite acting condition. As $r_D \to \infty$, $\hat{p}_D$ must remain bounded, however:

$$\lim_{x\to\infty} I_0(x) = \infty \tag{3.22}$$

To prevent $\hat{p}_D$ from going to infinity we set $c_1 = 0$.

$$\Rightarrow \hat{p}_D = c_2 K_0(r_D\sqrt{s}) \tag{3.23}$$

The inner boundary condition is:

$$\lim_{r_D \to 0} r_D \frac{\partial p_D}{\partial r_D} = -1 \tag{3.24}$$

$$\Rightarrow \lim_{r_D \to 0} \mathcal{L}\left\{ r_D \frac{\partial p_D}{\partial r_D} \right\} = \lim_{r_D \to 0} r_D \frac{\partial \hat{p}_D}{\partial r_D} = \mathcal{L}\{-1\} = -\frac{1}{s} \tag{3.25}$$

To differentiate the Bessel function we need the following recurrence relationship:

$$\frac{d}{dx}\left(x^{-n} K_n(x) \right) = -x^{-n} K_{n+1}(x) \tag{3.26}$$

Substituting (3.26) into (3.25) gives:

$$\lim_{r_D \to 0} r_D \frac{\partial}{\partial r_D}\left(c_2 K_0(r_D \sqrt{s}) \right) = -\frac{1}{s} \tag{3.27}$$

$$\Rightarrow \lim_{r_D \to 0} \left[-c_2 r_D \sqrt{s} K_1(r_D \sqrt{s}) \right] = -\frac{1}{s} \tag{3.28}$$

To evaluate the limit we can use the following limiting form of K_v for small arguments:

$$K_v(z) \approx \frac{1}{2}\Gamma(v)(\frac{1}{2}z)^{-v} \tag{3.29}$$

$$\Rightarrow \lim_{r_D \to 0} K_1(r_D \sqrt{s}) = \frac{1}{r_D \sqrt{s}} \tag{3.30}$$

$$\Rightarrow -c_2 \lim_{r_D \to 0} \left(r_D \sqrt{s} \frac{1}{r_D \sqrt{s}} \right) = -\frac{1}{s} \tag{3.31}$$

$$\Rightarrow c_2 = \frac{1}{s} \tag{3.32}$$

Finally we have the complete solution for $\hat{p}_D$:

$$\hat{p}_D(r_D, s) = \frac{1}{s} K_0(r_D \sqrt{s}) \tag{3.33}$$

Now invert $\hat{p}_D$ to find p_D. This can be achieved by recalling theorem 7:

$$\mathcal{L}\left\{ \int_0^t f(t)dt \right\} = \frac{1}{s}\mathcal{L}\{f(t)\} \tag{3.34}$$

To proceed the inverse transform of $K_0(r_S\sqrt{s})$ is required. Transform pair 117 from the handout gives the following:

$$\mathcal{L}^{-1}\{K_0(r_D\sqrt{s})\} = \frac{1}{2t_D} exp\left(\frac{-r_D^2}{4t_D}\right) \tag{3.35}$$

$$\Rightarrow p_D = \int_0^{t_D} \frac{1}{2t_D} exp\left(\frac{-r_D^2}{4t_D}\right) dt_D \tag{3.36}$$

This integral can be evaluated by using substitution:

$$u = \frac{r_D^2}{4t_D} \tag{3.37}$$

$$\Rightarrow t_D = \frac{r_D^2}{4u} \tag{3.38}$$

$$dt_D = \frac{-r_D^2}{4u^2} du \tag{3.39}$$

Equation (3.36) becomes:

$$p_D = \int_{\infty}^{\frac{r_D^2}{4t_D}} \frac{4u}{2r_D^2} exp(-u) \frac{-r_D^2}{4u^2} du = \frac{1}{2} \int_{\frac{r_D^2}{4t_D}}^{\infty} \frac{exp(-u)}{u} du \tag{3.40}$$

Now introduce the exponential integral, $Ei(x)$:

$$Ei(x) = -\int_{-x}^{\infty} \frac{e^{-u}}{u} du \tag{3.41}$$

(This definition follows Abramowitz and Stegun, “Handbook of Mathematical Functions”, Dover, 1970. Note that in some references $Ei(x)$ is denoted by $E_1(x)$ - be careful!)

$$\Rightarrow p_D = -\frac{1}{2} Ei\left(-\frac{r_D^2}{4t_D}\right) \tag{3.42}$$

Finally the answer is written in dimensional terms:

$$p = p_i + \frac{q\mu}{4\pi kh} Ei\left(-\frac{r^2\phi\mu c_t}{4kt}\right) \tag{3.43}$$

3.1.3 Late Time Behaviour of p_D

We can consider the late time behaviour of p_D by recalling theorem 5 and taking the limit of $\hat{p}_D$ as $s \to 0$.

$$\hat{p}_D(s) = \frac{1}{s} K_0(r_D\sqrt{s}) \tag{3.44}$$

The limit can be handled by using a series expansion for K_0:

$$K_0(x) = -(ln\frac{x}{2} + \gamma)I_0(x) + \frac{\frac{1}{4}z^2}{(1!)^2} + (1+\frac{1}{2})\frac{(\frac{1}{4}z^2)^2}{(2!)^2} + ... \tag{3.45}$$

$$\Rightarrow \lim_{x\to 0} K_0(x) = -\ln\left(\frac{x}{2}\right) - \gamma \tag{3.46}$$

where γ = Euler's constant, 0.5772.

$$\lim_{s\to 0} \hat{p}_D = -\frac{1}{s}\left(\ln r_D + \ln\sqrt{s} - \ln 2 + \gamma\right) \tag{3.47}$$

$$\Rightarrow \lim_{t\to\infty} p_D = \mathcal{L}^{-1}\left(-\frac{1}{s}(\ln r_D + \ln\sqrt{s} - \ln 2 + \gamma)\right) \tag{3.48}$$

Using transform pair 95 from the handout to invert the $ln\sqrt{s}$ term gives:

$$= -\ln r_D + \ln 2 - \gamma + \frac{1}{2}(\gamma + \ln t_D) \tag{3.49}$$

$$= \frac{1}{2}\left(\ln\frac{t_D}{r_D^2} + 0.80907\right) \tag{3.50}$$

3.2 Finite Well Radius Solution

The line source solution applies the constant flow rate condition as r tends to zero. This simplifies the solution process. However an analytical solution can also be obtained when the flow rate condition is applied at $r = r_w$. The governing equation and boundary conditions remain the same as the line source solution, except for the inner boundary condition which is now:

$$r_D\frac{\partial p_D}{\partial r_D}|_{r_D=1} = -1 \tag{3.51}$$

As before when the differential equation is written in Laplace space we have:

$$\Rightarrow \frac{\partial^2 \hat{p}_D}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial \hat{p}_D}{\partial r_D} - s\hat{p}_D = 0 \tag{3.52}$$

As before the general solution to the problem is:

$$\hat{p}_D = c_1 I_0(r_D\sqrt{s}) + c_2 K_0(r_D\sqrt{s}) \tag{3.53}$$

The solution must remain bounded as r tends to infinity so as before we set $c_1 = 0$:

$$\hat{p_D} = c_2 K_0(r_D\sqrt{s}) \tag{3.54}$$

Now consider the inner boundary condition. It requires that:

$$\frac{\partial}{\partial r_D}(c_2 K_0(r_D\sqrt{s}) = -\frac{1}{s} \tag{3.55}$$

$$\Rightarrow -c_2\sqrt{s}K_1(r_D\sqrt{s}) = -\frac{1}{s} \qquad r_D = 1 \tag{3.56}$$

$$\Rightarrow c_2 = \frac{1}{s^{\frac{3}{2}}K_1(\sqrt{s})} \tag{3.57}$$

The final solution for $\hat{p}_D$ is:

$$\hat{p}_D = \frac{K_0(r_D\sqrt{s})}{s^{\frac{3}{2}}K_1(\sqrt{s})} \tag{3.58}$$

3.2.1 Early Time Behaviour of p_D

The early time behaviour of p_D can be examined by considering the limit of $\hat{p}_D$ as $s \to \infty$. To do so the behaviour of the Bessel functions is required for large arguments.

$$K_v(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x}(1 + \frac{\mu - 1}{8x} + \frac{(\mu - 1)(\mu - 9)}{2!(8x)^2} + ... \tag{3.59}$$

where x is large and $\mu = 4v^2$.

$$\Rightarrow K_0(r_D\sqrt{s}) = \sqrt{\frac{\pi}{2r_D\sqrt{s}}}e^{-r_D\sqrt{s}} \tag{3.60}$$

and

$$\Rightarrow K_1(\sqrt{s}) = \sqrt{\frac{\pi}{2\sqrt{s}}} e^{-\sqrt{s}} \tag{3.61}$$

So at late time $\hat{p}_D$ is:

$$\hat{p}_D = \frac{1}{s^{\frac{3}{2}}} \sqrt{\frac{\pi}{2r_D\sqrt{s}}} e^{-r_D\sqrt{s}} \sqrt{\frac{2\sqrt{s}}{\pi}} e^{\sqrt{s}} \tag{3.62}$$

$$= \frac{1}{s^{\frac{3}{2}}} \frac{1}{\sqrt{r_D}} e^{-\sqrt{s}(r_D-1)} \tag{3.63}$$

The solution for p_D can be found using transform pair number 85 from the handout:

$$p_D(r_D, t_D) = \frac{1}{\sqrt{r_D}} \left\{ 2\sqrt{\frac{t_D}{\pi}} e^{-\frac{(r_D-1)^2}{4t_D}} - (r_D - 1) erfc \left(\frac{r_D - 1}{2\sqrt{t_D}} \right) \right\} \tag{3.64}$$

$$p_D(r_D = 1, t_D) = 2\sqrt{\frac{t_D}{\pi}} \tag{3.65}$$

3.2.2 Late Time Behaviour of p_D

To find the late time behaviour of p_D consider the limit of $\hat{p}_D$ at $s \to 0$. First consider the behaviour of the Bessel functions. As before:

$$\lim_{x\to 0} K_0(x) = -[ln(\frac{1}{2}x) + \gamma] \tag{3.66}$$

For small arguments $K_v(x)$ can be approximated by:

$$K_v(x) = \frac{1}{2}\Gamma(x)(\frac{1}{2}x)^{-v} \tag{3.67}$$

where $\Gamma(x+1) = x!$

$$\Rightarrow \lim_{x\to 0} K_1(x) = \frac{1}{2}(\frac{1}{2}x)^{-1} = \frac{1}{x} \tag{3.68}$$

Recall the solution for $\hat{p}_D$:

$$\hat{p}_D = \frac{K_0(r_D\sqrt{s})}{s^{\frac{3}{2}} K_1(\sqrt{s})} \tag{3.69}$$

The late time behaviour of p_D can be found by performing the following inverse transform:

$$p_D = \mathcal{L}^{-1}\left\{\frac{1}{s^{\frac{3}{2}}}\frac{-[\ln(\frac{1}{2}r_D\sqrt{s})+\gamma]}{\frac{1}{\sqrt{s}}}\right\} = \mathcal{L}^{-1} - \frac{1}{s}(\ln r_D + \ln\sqrt{s} - \ln 2 + \gamma) \quad (3.70)$$

$$= \frac{1}{2}\left(\ln\frac{t_D}{r_D^2} + 0.80907\right) \quad (3.71)$$

This is the same late time behaviour as the line source solution.

3.3 Constant Pressure Inner Boundary Condition

The previous two solutions have considered constant rate boundary conditions at the well. It is also possible to consider constant pressure boundary conditions. It becomes more convenient to define the dimensionless pressure, p_D, in terms of both the initial reservoir pressure, p_i, and the well pressure, p_w:

$$p_D = \frac{p_i - p}{p_i - p_w} \quad (3.72)$$

The dimensionless form of the pressure equation is as before:

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial p_D}{\partial r_D}\right) = \frac{\partial p_D}{\partial t_D} \quad (3.73)$$

For an infinite acting reservoir the boundary and initial conditions in dimensionless form are:

$$p_D(r_D, t_D) = 1 \qquad r_D = 1 \quad (3.74)$$

$$p_D(r_D \to \infty, t_D) = 0 \quad (3.75)$$

$$p_D(r_D, t_D = 0) = 0 \quad (3.76)$$

As before the general solution to this problem is:

$$\hat{p}_D = c_1 I_0(r_D\sqrt{s}) + c_2 K_0(r_D\sqrt{s}) \quad (3.77)$$

Again c_1 is set to zero to ensure the pressure remains finite as $r \to \infty$. The inner boundary condition is used to solve for c_2:

$$\hat{p}_D(r_D = 1) = c_2 K_0(\sqrt{s}) = \frac{1}{s} \quad (3.78)$$

$$\Rightarrow c_2 = \frac{1}{sK_0(\sqrt{s})} \tag{3.79}$$

$$\Rightarrow \hat{p}_D(r_D) = \frac{K_0(r_D\sqrt{s})}{sK_0(\sqrt{s})} \tag{3.80}$$

The inverse transform to solve this problem was provided by Van Everdingen and Hurst, "The Application of the Laplace Transformation to Flow Problems in Reservoirs", Petroleum Transactions AIME, 305-324, 1949 (see equation VI-26):

$$p_D(r_D, t_D) = \frac{2}{\pi}\int_0^\infty \frac{(1 - e^{-u^2 t_D})[J_0(u)Y_0(ur_D) - Y_0(u)J_0(ur_D)]}{u^2[J_0^2(u) + Y_0^2(u)]}du \tag{3.81}$$

3.3.1 Early Time Behaviour of the Flow Rates

Just as the early time behaviour of the pressure could be considered in the constant flow rate case, the behaviour of the flow rate can be examined for the constant pressure case:

$$q_D = -r_D\frac{\partial p_D}{\partial r_D} \tag{3.82}$$

$$\hat{q}_D = -r_D\frac{\partial \hat{p}_D}{\partial r_D} = -r_D\frac{\partial}{\partial r_D}\left(\frac{K_0(r_D\sqrt{s})}{sK_0(\sqrt{s})}\right) \tag{3.83}$$

Using the previously established expression for the derivative of K_0 gives:

$$\hat{q}_D = \frac{r_D}{\sqrt{s}}\frac{K_1(r_D\sqrt{s})}{K_0(\sqrt{s})} \tag{3.84}$$

The cumulative recovery is defined by:

$$Q_D = \int_0^{t_d} q_D dt_D \tag{3.85}$$

The Laplace transform of Q_D can be found readily by recalling theorem 7:

$$\hat{Q}_D = \frac{1}{s}\hat{q}_D = \frac{r_D}{s^{\frac{3}{2}}}\frac{K_1(r_D\sqrt{s})}{K_0(\sqrt{s})} \tag{3.86}$$

To consider the early time behaviour of the flow rates consider the limit of q_D as $s \to \infty$. The early time behaviour of the Bessel functions have already been established:

$$K_1(r_D\sqrt{s}) \approx \sqrt{\frac{\pi}{2\sqrt{s}r_D}}e^{-r_D\sqrt{s}} \tag{3.87}$$

$$K_0(\sqrt{s}) \approx \sqrt{\frac{\pi}{2\sqrt{s}}}e^{-\sqrt{s}} \tag{3.88}$$

$$\Rightarrow \hat{q}_D = \frac{r_D}{\sqrt{s}}\sqrt{\frac{\pi}{2\sqrt{s}r_D}}e^{-r_D\sqrt{s}}\sqrt{\frac{2\sqrt{s}}{\pi}}e^{\sqrt{s}} = \frac{\sqrt{r_D}}{\sqrt{s}}e^{-\sqrt{s}(r_D-1)} \tag{3.89}$$

q_D can now be found by using transform pair 84 from the tables:

$$q_D = \frac{\sqrt{r_D}}{\sqrt{\pi t_D}}e^{-\frac{(r_D-1)^2}{4t_D}} \tag{3.90}$$

Q_D (at early time) can be found by using transform pair 85:

$$Q_D = r_D\left\{2\sqrt{\frac{t_D}{\pi}}e^{-\frac{(r_D-1)^2}{4t_D}} - (r_D-1)erfc\left(\frac{r_D-1}{2\sqrt{t_D}}\right)\right\} \tag{3.91}$$

Note the similarity between this and Equation (3.64) (early time behaviour of the pressure for constant rate, finite radius well).

3.4 Bounded Reservoir Example

The previous examples have considered flow in infinite acting reservoirs. Linear boundaries in reservoirs with either constant pressure or constant flow rate wells can be created using superposition as discussed in Chapter 1. Consider a case with a constant flow rate and the well and a constant pressure at the outer boundary (at radius r_e). The boundary and initial conditions in dimensionless form are:

$$r_D\frac{\partial p_D}{\partial r_D} = -1 \qquad r_D = 1 \tag{3.92}$$

$$p_D = 0 \qquad r_D = \frac{r_e}{r_w} = r_{De} \tag{3.93}$$

$$p_D = 0 \qquad t_D = 0, \forall r_D \tag{3.94}$$

As before the general solution to this problem is:

$$\hat{p}_D = c_1 I_0(r_D\sqrt{s}) + c_2 K_0(r_D\sqrt{s}) \tag{3.95}$$

However in this example the reservoir is bounded so c_1 can not be set to zero by arguing that $\hat{p}_D$ must remain bounded as r_D tends to infinity. Instead the outer boundary condition requires:

$$c_1 I_0(r_{De}\sqrt{s}) + c_2 K_0(r_{De}\sqrt{s}) = 0 \tag{3.96}$$

The inner boundary conditions requires:

$$\frac{\partial}{\partial r_D}[c_1 I_0(r_D\sqrt{s}) + c_2 K_0(r_D\sqrt{s})] = -\frac{1}{s} \qquad r_D = 1 \tag{3.97}$$

The derivatives of the Bessel functions can be found from:

$$\frac{d}{dx}[x^{-n}K_n(x)] = -x^{-n}K_{n+1}(x) \tag{3.98}$$

$$\frac{d}{dx}[x^{-n}I_n(x)] = x^{-n}I_{n+1}(x) \tag{3.99}$$

Using these derivatives (3.97) becomes:

$$c_1\sqrt{s}I_1(\sqrt{s}) - c_2\sqrt{s}K_1(\sqrt{s}) = -\frac{1}{s} \tag{3.100}$$

$$\Rightarrow c_1 I_1(\sqrt{s}) - c_2 K_1(\sqrt{s}) = -\frac{1}{s^{\frac{3}{2}}} \tag{3.101}$$

The outer boundary condition requires:

$$c_1 I_0(r_{De}\sqrt{s}) + c_2 K_0(r_{De}\sqrt{s}) = 0 \tag{3.102}$$

Equations (3.101) and (3.102) can be solved for c_1 and c_2 to give:

$$c_1 = -\frac{1}{s^{\frac{3}{2}}}\frac{K_0(r_{De}\sqrt{s})}{K_0(r_{De}\sqrt{s})I_1(\sqrt{s}) + K_1(\sqrt{s})I_0(r_{De}\sqrt{s})} \tag{3.103}$$

$$c_2 = \frac{1}{s^{\frac{3}{2}}}\frac{I_0(r_{De}\sqrt{s})}{K_0(r_{De}\sqrt{s})I_1(\sqrt{s}) + K_1(\sqrt{s})I_0(r_{De}\sqrt{s})} \tag{3.104}$$

$$\Rightarrow \hat{p}_D = \frac{1}{s^{\frac{3}{2}}}\frac{I_0(r_{De}\sqrt{s})K_0(r_D\sqrt{s}) - K_0(r_{De}\sqrt{s})I_0(r_D\sqrt{s})}{K_0(r_{De}\sqrt{s})I_1(\sqrt{s}) + K_1(\sqrt{s})I_0(r_{De}\sqrt{s})} \tag{3.105}$$

The late time (steady state) behaviour of p_D can be determined by taking the limit of $\hat{p}_D$ as s tends to infinity:

$$p_D = ln\frac{r_D}{r_{De}} \tag{3.106}$$

3.5 Van Everdingen and Hurst

Van Everdingen and Hurst's 1949 paper was one of the first applications of Laplace transforms in petroleum reservoir engineering. One of the interesting results they proved was the following relationship between the pressure at a well (operating at constant flow rate) and the cumulative production (from a well operating at constant pressure):

$$s\hat{p_D}\hat{Q_D} = \frac{1}{s^2} \qquad r_D = 1 \tag{3.107}$$

Van Everdingen and Hurst also demonstrated how to add wellbore storage to a problem:

$$\hat{p}_D = \frac{s\hat{p}_{Dxx}}{s(1 + sc_D s\hat{p}_{Dxx})} \tag{3.108}$$

3.6 Incorporating Storage and Skin

Wellbore storage means that even if a well is produced at constant flow rate the flow from the reservoir into the wellbore may be transient. The additional flow can come from either the expansion of fluid in the wellbore or a changing liquid level in the tubing.

3.6.1 Fluid Expansion

$$q_{total} = q_{reservoir} + q_{expansion} \tag{3.109}$$

The amount of flow from fluid expansion is defined by:

$$q_{expansion} = \frac{\partial V_w}{\partial t} = \frac{\partial V_w}{\partial p}\frac{\partial p}{\partial t} = c_w V_w \frac{\partial p}{\partial t} \tag{3.110}$$

The relevant storage coefficient, C is defined by:

$$C = c_w V_w \tag{3.111}$$

3.6.2 Falling Liquid Level

If a well is completed without a packer there may be liquid in the annulus. This fluid may be produced when the bottom hole pressure is lowered. The relevant storage coefficient, C is defined by:

$$C = \frac{A_w}{\rho g} \tag{3.112}$$

3.6.3 Laplace Space Solution

If the Laplace space solution for a problem which has no storage and no skin is given by $\hat{p}_{Dxx}$ then solution for a problem with a skin, S, and storage, c_D is:

$$\hat{p}_D = \frac{s\hat{p}_{Dxx} + S}{s[1 + sc_D(s\hat{p}_{Dxx} + S)]} \tag{3.113}$$

The dimensionless storage c_D is defined by:

$$C_D = \frac{5.615C}{2\pi\phi c_t h r_w^2} \tag{3.114}$$

3.7 Incorporating Dual Porosity

In dual porosity reservoirs flow occurs in both the matrix and in the fractures. Their are two important parameters in dual porosity reservoirs:

$$\omega = \frac{\phi_f c_{tf}}{\phi_f c_{tf} + \phi_m c_{tm}} \qquad 0 < \omega < 1 \tag{3.115}$$

$$\lambda = \alpha \frac{k_m}{k_f} r_w^2 \qquad 10^{-10} < \lambda < 10^{-3} \tag{3.116}$$

where α depends on the fracture configuration e.g. sugar cube model
The governing equation in Laplace space for a dual porosity reservoir is:

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D \frac{\partial \hat{p}_D}{\partial t_D}\right) - sf(s)\hat{p} = 0 \tag{3.117}$$

where

$$f(s) = \frac{s\omega(1-\omega) + \lambda}{s(1-\omega) + \lambda} \tag{3.118}$$

For the case of an infinite acting reservoir with a finite radius the solution for $\hat{p}_D$ is:

$$\hat{p}_D = \frac{K_0(\sqrt{sf(s)})}{s\sqrt{sf(s)}K_1(\sqrt{sf(s)})} \tag{3.119}$$

(for more details see “Well Test Analysis”, Rajagopal Raghavan, Englewood Cliffs, 1993)

3.8 Bourgeois and Horne

Marcel Bourgeois completed an MS degree in the Department of Petroleum Engineering in 1992 (”Well Test Interpretation Using Laplace Space Type Curves”). This was a key contribution to the use of Laplace transforms in well testing. The solutions to many well testing problems (beyond the examples presented here) are known in Laplace space. As part of his study Bourgeois defined a quantity known as the Laplace pressure, $s\hat{p}$. When plotted against $1/s$ this has similar behaviour as the real pressure

Bourgeois showed that instead of performing parameter estimation using nonlinear regression in real space the matching could be achieved efficiently and effectively in Laplace space. The efficiency lies in the removal of the need for a numerical inverse transform to evaluate the performance of set of paramter estimates. In Bourgeois’ examples fewer iterations were needed when matching in Laplace space than in real space.

3.9 Heat Transfer

Laplace transforms are also useful in other petroleum engineering engineering problems. During my undergraduate research project I studied heat transfer from a buried pipeline which connects an offshore platform to onshore production facilities. Heat transfer was a particular concern because the fluids could form solid hydrates if the temperature fell below a critical level.

The pipe is buried below the sea floor. This makes the computational domain semi-infinite. However in the part of the study that used Laplace transforms an “effective cylinder” configuration was used as shown in Figure 3.1.

Two energy balance equations are required in this problem. The first governs the temperature distribution in the pipe surrounds:

$$k_e \nabla^2 T_e = \rho_e c_e \frac{\partial T_e}{\partial t} \tag{3.120}$$

The second energy balance governs the fluid in the pipe which is flowing

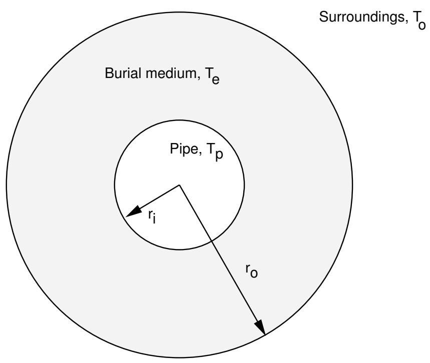

Figure 3.1: Configuration of pipe and surrounds

at velocity, U:

$$\rho_p c_p \frac{\partial T_p}{\partial t} + \rho_p c_p U \frac{\partial T_p}{\partial x} = -\frac{2}{r_i} q \tag{3.121}$$

where q is the flux of heat through the pipe wall:

$$q = -k_e \frac{\partial T_e}{\partial r}|_{r=r_i} \tag{3.122}$$

The solution procedure involved solving for the Laplace transform of T_e in terms of T_p. This expression was then substituted into the equation governing the Laplace transform of T_p. Finally the $\hat{T}_p$ was solved for.

An example of this solution given below for the case of the pipeline heating up at start-up:

$$\hat{T}_p(x,s) = \frac{T_i - T_o}{s} exp(-(\frac{s}{U} + \frac{2k_e\sqrt{s\kappa}\gamma}{r_i\rho_p c_p U\theta})x) + \frac{T_o}{s} \tag{3.123}$$

where

$$\theta = K_0(\sqrt{s\kappa}r_i)I_0(\sqrt{s\kappa}r_o) - K_o(\sqrt{s\kappa}r_o)I_0(\sqrt{s\kappa}r_i) \tag{3.124}$$

$$\kappa = \frac{\rho_e c_e}{k_e} \tag{3.125}$$

$$\gamma = K_0(\sqrt{s\kappa}r_0)I_1(\sqrt{s\kappa}r_i) + K_1(\sqrt{s\kappa}r_i)I_0(\sqrt{s\kappa}r_o) \tag{3.126}$$

The transient pipeline temperature distribution could be found by numerically inverting the expression for $\hat{T}_p$. The results from this approach were much closer to field test results than results from a large finite difference simulation. Since the numerical inverse can be computed very quickly many more cases could be considered to assess the sensitivities to various parameters in the model.

3.10 Numerical Inversion of Laplace Transforms

The inverse Laplace transform can also be written as an integral:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+\infty} e^{st}F(s)ds \tag{3.127}$$

where γ is chosen in such away that any singularities in $F(s)$ are avoided. The contour the integral is performed over is known as the Bromwich contour. When an inverse transform is required that can't be found from tables this integral is usually evaluated numerically.

The most commonly used algorithm for numerical inversion of Laplace transforms is the Stehfest algorithm (Communications of the Association for Computing Machinery, algorithm 368). To invert $\hat{f}(s)$ the following summation is performed:

$$f(t) = \frac{\ln 2}{t}\sum_{i=1}^{N} V_i \hat{f}\left(\frac{\ln 2}{t}i\right) \tag{3.128}$$

where

$$V_i = (-1)^{\frac{N}{2+i}} \sum_{k=\left(\frac{i+1}{2}\right)}^{min(i,N/2)} \frac{k^{\frac{N}{2}}(2k)!}{(N/2-k)!k!(k-1)!(i-k)!(2k-i)!} \tag{3.129}$$

Theoretically the accuracy of $f(t)$ increases as N increases. However in practice the V_i grow quickly in magnitude with N and round-offs errors are amplified. Usually $N = 8$ is used in numerical inversions. This means that for every value of $f(t)$ required 8 values of $\hat{f}(s)$ are required.

The Stehfest algorithm works well for smooth functions but has difficulties for oscillatory functions and functions with discontinuities. Oscillatory functions can be inverted if their wavelength is large with respect to half the width of the peaks. Stehfest tested his algorithm on 50 functions and reported errors of only 0.1%.

There are other algorithms available for the numerical inversion of Laplace transforms. The Talbot algorithm (J. Inst. Math. Appl., Jan. 1979, pg 97-120) is one of the most accurate and widely applicable. Other algorithm seek to reuse previously evaluated $\hat{f}(s)$ values in the evaluation of subsequent $f(t)$ values.

3.11 Summary

This chapter has outlined several petroleum engineering applications of Laplace transforms including:
- the line source solution
- the finite well radius solution
- constant well pressure solution
- bounded reservoir solution

Laplace transforms are attractive for these problems because storage, skin and dual porosity behaviour can be added readily to the solutions in Laplace space. The Laplace space solution can also be efficiently incorporated into nonlinear regression routines.

A heat transfer case study was discussed to demonstrate how the use of Laplace space solutions can complement numerical methods. The solution that study developed approximated the physics and the geometry of the problem but ran very quickly and could be used for sensitivity analyses. It ultimately reproduced the field results more accurately than the numerical model results.

Chapter 4

Fourier Transforms

Like the Laplace transform the Fourier transform is also an integral transform. When viewed in the context of signal processing the application of the Fourier transform takes a function from real-space to frequency-space (see later examples). The Fourier transform is defined by:

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi xs} dx \tag{4.1}$$

The inverse transform is defined in a similar manner:

$$f(x) = \int_{-\infty}^{\infty} F(s) e^{i2\pi xs} ds \tag{4.2}$$

We will also use the notation $\tilde{f}(s)$ for $F(s)$. The Fourier transform exists if $f(x)$ and $f'(x)$ are at least piecewise continuous and the following integral exists:

$$\int_{-\infty}^{\infty} |f(x)| dx \tag{4.3}$$

There are also some alternative definitions:

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-ixs} dx \tag{4.4}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{ixs} ds \tag{4.5}$$

and

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixs} dx \tag{4.6}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{ixs} ds \tag{4.7}$$

Example:

$$f(x) = e^{-\pi x^2} \tag{4.8}$$

$$F(s) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-i2\pi s x} dx \tag{4.9}$$

$$= \int_{-\infty}^{\infty} e^{-\pi(x^2 + i2xs)} dx \tag{4.10}$$

$$= \int_{-\infty}^{\infty} e^{-\pi(x^2 + i2xs - s^2 + s^2)} dx \tag{4.11}$$

$$= \int_{-\infty}^{\infty} e^{-\pi(x+is)^2 - \pi s^2} dx \tag{4.12}$$

$$= e^{-\pi s^2} \int_{-\infty}^{\infty} e^{-\pi(x+is)^2} dx \tag{4.13}$$

$$= e^{-\pi s^2} \int_{-\infty}^{\infty} e^{-\pi \xi^2} dx \tag{4.14}$$

where $\xi = x + is$ The integral in (4.14) is known to be 1.0 so we have:

$$F(s) = e^{-\pi s^2} \tag{4.15}$$

The Fourier transform relates a function in real space (either time or distance) to a function in frequency space. This can be seen by recalling:

$$e^{i2\pi xs} = \cos(2\pi xs) + i\sin(2\pi xs) \tag{4.16}$$

Now consider the inverse transform:

$$f(x) = \int_{-\infty}^{\infty} F(s) e^{i2\pi xs} ds \tag{4.17}$$

This integral shows that the Fourier transform breaks a function $f(x)$ into a sum of sines and cosines with frequency s. (Recall the frequency of $f(kx)$ is $\frac{|k|}{2\pi}$). The ammplitude associated with any given frequency is given by $F(s)$.
Example:
Consider $f = \cos(\pi x)$. The Fourier transform of f is:

$$F(s) = \frac{1}{2}\delta(-\pi + 2\pi s) + \frac{1}{2}\delta(\pi + 2\pi s) \tag{4.18}$$

i.e.

$$f(x) = \frac{1}{2}(\cos(\pi x) + i\sin(\pi x) + \cos(-\pi x) + i\sin(-\pi x)) \tag{4.19}$$

$$= \frac{1}{2}(\cos(\pi x) + \cos(\pi x)) = \cos(\pi x) \tag{4.20}$$

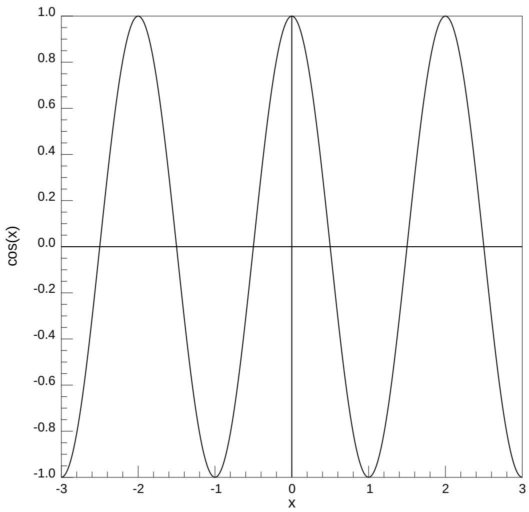

Figure 4.1: $f(x) = cos(\pi x)$

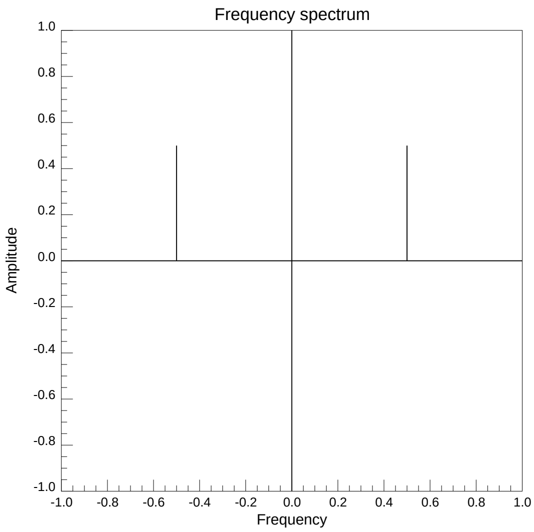

Figure 4.2: Frequency spectrum of $f(x) = cos(\pi x)$

4.1 Fourier Transform Theorems

4.1.1 Theorem 1 - Linearity

$$F(f(x)+g(x)) = F(f(x)) + F(g(x)) \tag{4.21}$$

4.1.2 Theorem 2 - Shift Theorem

If

$$F(f(x)) = F(s) \tag{4.22}$$

then

$$F(f(x-a)) = e^{-i2\pi sa}F(s) \tag{4.23}$$

Proof:

$$F(f(x-a)) = \int_{-\infty}^{\infty} f(x-a)e^{-i2\pi sx}dx \tag{4.24}$$

$$= \int_{-\infty}^{\infty} f(x-a)e^{-i2\pi s(x-a)-i2\pi sa}d(x-a) \tag{4.25}$$

$$= e^{-i2\pi sa}\int_{-\infty}^{\infty} f(x-a)e^{-i2\pi s(x-a)}d(x-a) \tag{4.26}$$

$$= e^{-i2\pi sa}F(s) \tag{4.27}$$

4.1.3 Theorem 3 - Similarity Theorem

If

$$F(f(x)) = F(s) \tag{4.28}$$

$$F(f(ax)) = \frac{1}{|a|}F\left(\frac{s}{a}\right) \tag{4.29}$$

4.1.4 Theorem 4 - Convolution Theorem

If

$$F(f(x)) = F(s) \tag{4.30}$$

and

$$F(g(x)) = G(s) \tag{4.31}$$

then

$$F(f(x)*g(x)) = F(s)G(s) \tag{4.32}$$

4.1.5 Theorem 5 - Parseval's theorem

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds \tag{4.33}$$

4.1.6 Theorem 6 - Derivatives

$$F(f^n(x)) = (i2\pi s)^n F(s) \tag{4.34}$$

Note that this assumes the values of the derivatives vanish at $\pm\infty$.

4.2 Fourier Sine and Cosine Transforms

The Fourier transform is defined as:

$$F(f(x)) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi sx} dx \tag{4.35}$$

$$= \int_{-\infty}^{\infty} f(x)[cos(-2\pi sx) + i\ sin(-2\pi sx)]dx \tag{4.36}$$

Now consider a case where $f(x)$ is the sum of an even and an odd function, $f_e(x)$ and $f_o(x)$.
Recall for an odd function:

$$f_o(-x) = -f_o(x) \tag{4.37}$$

$sin(x)$ is an example of an odd function.
For an even function:

$$f_e(-x) = f_e(x) \tag{4.38}$$

$cos(x)$ is an example of an even function.
With $f(x)$ defined as the sum of an even and odd function the Fourier transform of $f(x)$ becomes:

$$F(f(x)) = \int_{-\infty}^{\infty} (f_e(x)+f_o(x))cos(2\pi sx)dx - i\int_{-\infty}^{\infty} (f_e(x)+f_o(x))sin(2\pi sx)dx \tag{4.39}$$

Now take account of the way products of even and odd functions behave:

$$f_e(x)g_e(x) = h_e(x) \tag{4.40}$$

$$f_o(x)g_o(x) = h_e(x) \tag{4.41}$$

$$f_o(x)g_e(x) = h_o(x) \tag{4.42}$$

Also note the following integral:

$$\int_{-\infty}^{\infty} f_o(x)dx = 0 \tag{4.43}$$

Now substitute these relationships into (4.39):

$$F(f(x)) = \int_{-\infty}^{\infty} f_e(x)\cos(2\pi sx)dx - i\int_{-\infty}^{\infty} f_o(x)\sin(2\pi sx)dx \tag{4.44}$$

$$= 2\int_{0}^{\infty} f_e(x)\cos(2\pi sx)dx - 2i\int_{0}^{\infty} f_o(x)\sin(2\pi sx)dx \tag{4.45}$$

The fact that the Fourier transform splits into two terms (sine and cosine) motivates the definition of the sine and cosine transforms:

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)cos(xs)dx = F_c(s) \tag{4.46}$$

$$F_c^{-1}(F_c(s)) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(s)cos(xs)ds = f(x) \tag{4.47}$$

$$F_c(f'') = -s\sqrt{\frac{2}{\pi}}f'(0) - s^2F_c(s) \tag{4.48}$$

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)sin(xs)dx = F_s(s) \tag{4.49}$$

$$F_s^{-1}(F_s(s)) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(s)sin(xs)ds = f(x) \tag{4.50}$$

$$F_s(f'') = s\sqrt{\frac{2}{\pi}}f(0) - s^2F_s(s) \tag{4.51}$$

Use of sine and cosine transforms simplifies the transform procedure when transforming even and odd functions. The sine and cosines transforms can be used in place of the full Fourier transform for problems with:
- semi-infinite domains
- differential equation that have even orders of derivatives
- either f of f' specified at the boundary

4.3 Example 1: 1D Pressure Diffusion

Consider a one dimensional problem governed by:

$$\frac{\partial^2 p_D}{\partial x_D^2} = \frac{\partial p_D}{\partial t_D} \tag{4.52}$$

The boundary conditions are:

$$p_D(x_D = 0, t_D) = 1 \tag{4.53}$$

$$p_D(x_D \to \infty, t_D) = 0 \tag{4.54}$$

$$p_D(x_D, t_D =) = 0 \tag{4.55}$$

Since the pressure and not the pressure derivative is set on the boundary use the sine transform. The choice of transform is made according to equations (4.48) and (4.51) which relate the transform of the second derivative to the boundary conditions.
First transform the differential equation (in space):

$$s\sqrt{\frac{2}{\pi}} p_D(x_D = 0, t_D) - s^2 \tilde{p}_D = \frac{\partial \tilde{p}_D}{\partial t_D} \tag{4.56}$$

$$\Rightarrow \frac{\partial \tilde{p}_D}{\partial t_D} + s^2 \tilde{p}_D = \sqrt{\frac{2}{\pi}} \tag{4.57}$$

This equation can be solved using the integrating factor method:

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{4.58}$$

$$ye^{\int P dx} = \int Q e^{\int P dx} dx + C \tag{4.59}$$

$$\Rightarrow \tilde{p}_D e^{s^2 t_D} = \int_0^{t_D} \sqrt{\frac{2}{\pi}} s e^{s^2 \tau} d\tau + C \tag{4.60}$$

$$\tilde{p}_D = \int_0^{t_D} \sqrt{\frac{2}{\pi}} s e^{-s^2 (t_D - \tau)} d\tau \tag{4.61}$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{s} (1 - e^{-s^2 t_D}) \tag{4.62}$$

Now invert to find p_D:

$$F_s^{-1}(\tilde{p}_D) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{p}_D \sin(sx_D) ds \tag{4.63}$$

$$= \frac{2}{\pi} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{1}{s} (1 - e^{-s^2 t_D}) \sin(sx_D) ds \tag{4.64}$$

$$= 1 - Erf\left(\frac{x_D}{\sqrt{4t_D}}\right) = Erfc\left(\frac{x_D}{\sqrt{4t_D}}\right) \tag{4.65}$$

where $Erf(z)$ and $Erfc(z)$ are the error function and the complimentary error function defined by:

$$Erf(z) = \frac{2}{\sqrt{\pi}} = \int_0^z e^{-t^2} dt \tag{4.66}$$

$$Erfc(z) = 1 - Erf(z) \tag{4.67}$$

4.4 Example 2: Heat Equation

Consider the diffusion of heat in a one-dimensional bar. We'll consider an infinitely long bar so the full Fourier transform is required. The governing equation is:

$$\kappa^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \tag{4.68}$$

The boundary conditions are:

$$T(x \pm \infty, t) = T'(x \pm \infty) = 0 \tag{4.69}$$

The initial condition is a prescribed temperature that varies in space:

$$T(x, t = 0) = T_0(x) \tag{4.70}$$

First consider the Fourier transform of the spatial derivatives:

$$F(f'(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-isx} f'(x) dx \tag{4.71}$$

$$= \frac{1}{\sqrt{2\pi}} f(x) e^{-isx}|_{-\infty}^\infty - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty (-is) f(x) e^{-isx} dx \tag{4.72}$$

Since $f(x)$ vanishes at $\pm\infty$

$$= (is)F(f(x)) \tag{4.73}$$

Similar arguments require $f'(x)$ vanishes at $\pm\infty$. The transformed differential equation is:

$$\frac{\partial \tilde{T}}{\partial t} + \kappa^2 s^2 \tilde{T} = 0 \tag{4.74}$$

$$\Rightarrow \tilde{T} = c_1 e^{-(\kappa s)^2 t} \tag{4.75}$$

Now consider the initial condition:

$$\tilde{T}(t=0) = \tilde{T}_0 = c_1 \tag{4.76}$$

$$\Rightarrow \tilde{T} = \tilde{T}_0 e^{-(\kappa s)^2 t} \tag{4.77}$$

Now invert to find T:

$$T = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \tilde{T}_0 e^{-(\kappa s)^2 t} ds \tag{4.78}$$

The transform of T_0 is:

$$\tilde{T}_0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} T_0(\lambda) e^{-is\lambda} d\lambda \tag{4.79}$$

$$\Rightarrow T = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} \int_{-\infty}^{\infty} e^{-is\lambda} T_0(\lambda) e^{-(\kappa s)^2 t} d\lambda ds \tag{4.80}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} T_0(\lambda) \int_{-\infty}^{\infty} e^{-is(\lambda - x) - (\kappa s)^2 t} ds d\lambda \tag{4.81}$$

Finally we have an expression for T in terms of T_0:

$$T = \frac{1}{\sqrt{4\kappa^2 \pi t}} \int_{-\infty}^{\infty} T_0(\lambda) e^{-\frac{(\lambda - x)^2}{4\kappa^2 t}} d\lambda \tag{4.82}$$

4.5 Example 3: Elliptic Problem

Consider a steady-state problem in a two-dimensional semi-infinite domain governed by:

$$\nabla^2 p = 0 \tag{4.83}$$

The boundary conditions are:

$$p(0, y) = 0 \tag{4.84}$$

$$p(x \to \infty, y) = 0 \tag{4.85}$$

$$p(x, 0) = f(x) \tag{4.86}$$

$$p(x, a) = 0 \tag{4.87}$$

Since the domain is semi-infinite and the pressure is specified on the boundary we will use the sine transform to transform the differential equation:

$$F_s\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) = 0 \tag{4.88}$$

$$\Rightarrow s\sqrt{\frac{2}{\pi}}p(0, y) - s^2\tilde{p} + \frac{\partial^2 \tilde{p}}{\partial y^2} = 0 \tag{4.89}$$

$$\Rightarrow \frac{\partial^2 \tilde{p}}{\partial y^2} - s^2\tilde{p} = 0 \tag{4.90}$$

This equation can be solved for $\tilde{T}$ to give:

$$\tilde{T} = c_1 cos(isy) + c_2 sin(isy) \tag{4.91}$$

Now use the boundary conditions to determine c_1 and c_2:

$$F_s(p(x, y = 0)) = F_s(f(x)) = F_1 \tag{4.92}$$

$$F_s(p(x, y = a)) = 0 \tag{4.93}$$

After some algebra we can show:

$$\tilde{p} = F_1 \frac{sinh(s(a - y))}{sinh(sa)} \tag{4.94}$$

where

$$sinh(z) = -isin(iz) \tag{4.95}$$

Now invert to find p:

$$p = \sqrt{\frac{2}{\pi}} \int_0^\infty F_1 \frac{sinh(s(a - y))}{sinh(sa)} sin(xs) ds \tag{4.96}$$

where

$$F_1 = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\lambda) sin(s\lambda) d\lambda \tag{4.97}$$

Substituting F_1 into the expression for p gives:

$$\frac{2}{\pi} \int_0^\infty \int_0^\infty f(\lambda) \frac{sinh(s(a-y))}{sinh(sa)} sin(s\lambda) sin(sx) d\lambda ds \tag{4.98}$$

$$= \frac{y}{\pi} \int_0^\infty f(\lambda) \left(\frac{1}{y^2 + (x-\lambda)^2} - \frac{1}{y^2 + (x+\lambda)^2} \right) d\lambda \tag{4.99}$$

4.6 Radial Problems

All the examples presented have been for linear problems. Radial problems are often of more interest to petroleum engineers. Is the Fourier transform helpful in these cases?

$$\frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial p_D}{\partial r_D} = \frac{\partial p_D}{\partial t_D} \tag{4.100}$$

- The sine and cosine transforms won't work because the pressure equation in radial coordinates includes both even and odd orders of derivatives.
- The full Fourier transform is a candidate - consider applying it in space. When transforming the spatial derivatives we will require the behaviour of the pressure at $\pm\infty$. Ideally the pressure and it's first derivative would vanish at $\pm\infty$. This may be the case at $r = +\infty$ however it much harder to make that claim at $r = -\infty$. Applying the full Fourier transform in time is another option. However transforming the time derivative requires the behaviour of the pressure at $t = -\infty$. This is not such a problem since it is likely $p = p_i$ would be suitable. However now the boundary conditions become time dependent if the flow begins at $t = 0$.

The Hankel transform is better suited to radial problems.

$$H_v(\lambda) = \int_0^\infty r J_v(\lambda r) f(r) dr \tag{4.101}$$

$$f(r) = \int_0^\infty \lambda J_v(\lambda r) H_v(\lambda) d\lambda \tag{4.102}$$

We'll discuss this in a later section.

4.7 Inverting Fourier Transforms Numerically

Unlike the Laplace transform there is no special algorithm (like the Stehfest algorithm) required to invert Fourier transforms numerically. Since the limits of the inversion integral are real standard numerical integration routines can be used to evaluate the integral. The Stehfest routine required eight evaluations of the integrand to determine a numerical inverse Laplace transform at a given time. Many more evaluations of the integrand may be required when inverting a Fourier transform. If the Fourier transform is being applied to discrete data (discrete Fourier transform) instead of a function there are formal algorithms that can be used to perform the inversion.

4.8 Discrete Fourier Transforms

Instead of considering the Fourier transform of a continuous function consider a set of sampled points:

$$f_k = f(x_k), \qquad x_k = k\Delta, \qquad k = 0, 1, 2, ..., N-1 \tag{4.103}$$

We will seek estimates of the Fourier transform at discrete values of s:

$$s_n = \frac{n}{N\Delta} \tag{4.104}$$

The Fourier transform at s_n is:

$$F(s_n) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x s_n} dx \approx \sum_{k=0}^{N-1} f_k e^{-2\pi i s_n x_k} \Delta = \Delta \sum_{k=0}^{N-1} f_k e^{-2\pi i k n/N} \tag{4.105}$$

Similarly the inverse transform is:

$$f_k = \frac{1}{N} \sum_{n=0}^{N-1} F(s_n) e^{2\pi i k n/N} \tag{4.106}$$

For instance if we have 100 data points sampled from the following function over x[0,1]:

$$f(x) = sin(20\pi x) + noise \tag{4.107}$$

The function f(x), shown in Figure 4.3, is quite noisey. However by taking the Fourier transform, (Figure 4.4) we can extract the original sine wave

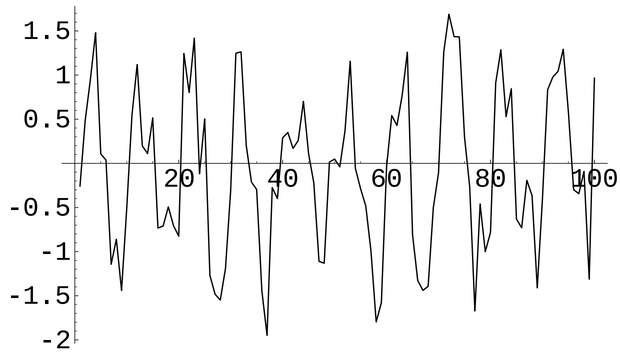

Figure 4.3: Noisy signal

quite easily. The Fourier transform shows two distinct spikes, one at the $n = 10$ and one at $n = 90$. These correspond to frequencies of ± 10 i.e. the frequency of the original sine wave. The first $N/2$ values of the Fourier transform correspond to frequencies of $0 < f < f_{max}$. The second $N/2$ values of the Fourier transform correspond to the frequencies $-f_{max} < f < 0$. Note that the value at $n = N/2$ corresponds to both $f = f_{max}$ and $-f_{max}$.

Note the discrete Fourier transform (DFT) only considers a finite range of frequencies because it uses a finite number of s_n. If there are frequencies beyond this present in the true Fourier transform and effect known as aliasing occurs. As shown in Figure 4.5 aliasing occurs when frequencies beyond the range of the chosen set of s_n are present. Aliasing "folds" these frequencies back into the computed Fourier transform. Aliasing can be avoided by filtering the function before it is sampled.

4.9 Fast Fourier Transforms

The discrete Fourier transform, as it was presented in the previous section, requires $O(N^2)$ operations to compute. In fact the discrete Fourier transform can be computed much more efficiently than that ($O(Nlog_2N)$ operations) by using the fast Fourier transform (FFT).

The concept of the FFT is outlined below (based on "Numerical Recipes in C"). A more specialized text should be consulted for details of the implementation. The FFT arises by noting that a DFT of length N can be

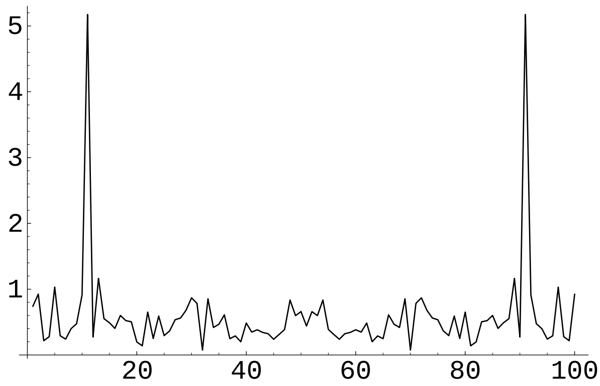

Figure 4.4: Fourier transform

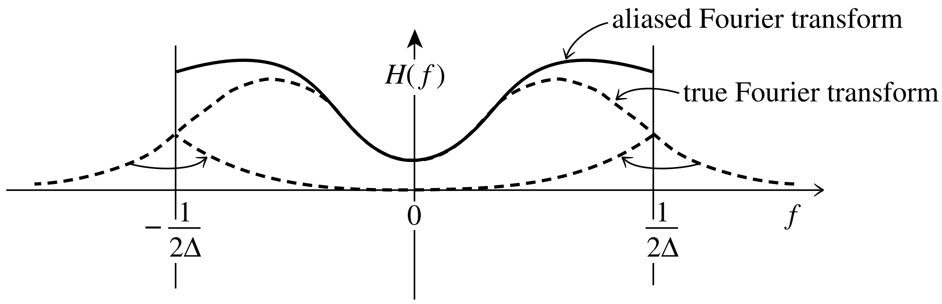

Figure 4.5: Aliasing effect (from “Numerical Recipes in C”, Cambridge University Press

written as the sum of two Fourier transforms each of length $N/2$. One of these transforms is formed from the even-numbered points of the original N, the other from the odd-numbered points.

$$F(s_k) = \sum_{j=0}^{N-1} e^{-2\pi ijk/N} f_j \tag{4.108}$$

$$= \sum_{j=0}^{N/2-1} e^{-2\pi ik(2j)/N} f_{2j} + \sum_{j=0}^{N/2-1} e^{-2\pi ik(2j+1)/N} f_{2j+1} \tag{4.109}$$

$$= \sum_{j=0}^{N/2-1} e^{-2\pi ikj/(N/2)} f_{2j} + W^k \sum_{j=0}^{N/2-1} e^{-2\pi ikj/(N/2)} f_{2j+1} \tag{4.110}$$

where

$$W = e^{-2\pi i/N} \tag{4.111}$$

$$\Rightarrow = F^e(s_k) + W^k F^o(s_k) \tag{4.112}$$

This expansion can be performed recursively i.e. a transform a length $N/2$ can be written as the sum of two transforms of length $N/4$ etc.

Chapter 5

Hartley Transforms and Hankel Transforms

5.1 Hartley Transforms

The Hartley transform was first described by Bracewell in 1984 (Bracewell, R.N. "The Fast Hartley Transform", Proceedings of the IEEE, v 72, n 8, p 1010, 1984). It is an alternative to the Fourier transform. The transform is defined by:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)[cos(2\pi st) + sin(2\pi st)]dt \tag{5.1}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)[cos(2\pi st) + sin(2\pi st)]ds \tag{5.2}$$

The notation $cas(2\pi st)$ is sometimes used:

$$cas(2\pi st) = cos(2\pi st) + sin(2\pi st) \tag{5.3}$$

Like the Fourier transform alternative definitions are possible:

$$F(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)[cos(2\pi st) + sin(2\pi st)]dt \tag{5.4}$$

$$f(t) = \int_{-\infty}^{\infty} F(s)[cos(2\pi st) + sin(2\pi st)]ds \tag{5.5}$$

Like the Fourier transform the Hartley transform maps a real signal into a function of frequency. Unlike the Fourier transform this function of frequency is real not complex. If the transform is being computed analytically this may make the algebra involved easier. If the transform is being performed numerically there is a considerable decrease in the amount of computation required. A complex multiplication or division requires four operations and a complex addition or subtraction requires a two operations. Also real data arrays require only half the storage of complex data arrays. So the Fast Hartley transform (FHT) requires considerably less memory and CPU time than the Fast Fourier Transform (FFT). Another attractive feature of the Hartley transform is that the transform and its inverse are symmetrical so the same piece of code can be used to compute the transform and the inverse.

5.2 Hankel Transforms

As we discussed in an earlier chapter the Fourier transform is not particulary appropriate for the spatial domain of semi infinite (or bounded) radial problems because we must make assumptions about the behaviour of the pressure at $\pm\infty$. The Hankel transform is a more suitable choice:

$$F_v(\lambda) = \int_0^\infty r J_v(\lambda r) f(r) dr \tag{5.6}$$

$$f(r) = \int_0^\infty \lambda J_v(\lambda r) F_v(\lambda) d\lambda \tag{5.7}$$

The Hankel transform is in fact a family of transforms, depending on the order v of the Bessel function involved. For our applications we will consider Bessel functions of order zero.

5.2.1 Properties of Hankel Transforms

Parseval's Theorem

There is no direct analogue to the convolution theorem for Hankel transforms however the following theorem can be readily proved:

$$\int_0^\infty F_v(\lambda) G_v(\lambda) \lambda d\lambda = \int_0^\infty f(x) g(x) x dx \tag{5.8}$$

Derivatives

Consider the Hankel transform of $g(x) = f'(x)$.

$$G_v(\lambda) = \int_0^\infty f'(x) J_v(\lambda x) x dx \tag{5.9}$$

$$= [x f(x) J_v(\lambda x)]_0^\infty - \int_0^\infty f(x) \frac{d}{dx}[x J_v(\lambda x)] dx \tag{5.10}$$

Assume that $f(x)$ is such that the first term is zero. Now consider the derivatives of the Bessel functions:

$$\frac{d}{dx}[x J_v(\lambda x)] = \frac{\lambda x}{2v}[(v+1) J_{v-1}(\lambda x) - (v-1) J_{v+1}(\lambda x)] \tag{5.11}$$

$$\Rightarrow G_v(\lambda) = -\lambda[\frac{v+1}{2v} F_{v-1}(\lambda) - \frac{v-1}{2v} F_{v+1}(\lambda)] \tag{5.12}$$

Note that $v = 0$ is a special case.

Bessel's Equation

One of the most useful features of the Hankel transform is what happens when it is applied to Bessel's equation. If $f(x)$ is an arbitrary function consider the transform of:

$$g(x) = \frac{d^2}{dx^2} f(x) + \frac{1}{x}\frac{d}{dx} f(x) - \frac{v^2}{x^2} f(x) \tag{5.13}$$

$$G_v(\lambda) = -\lambda^2 F_v(\lambda) \tag{5.14}$$

Note that the terms in the radial diffusivity equation, $\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r}$, are an instance of Bessels equations so in radial (no θ, z) coordinates:

$$F_0(\nabla^2 p) = -\lambda^2 F_0(\lambda) \tag{5.15}$$

For more information on Hankel transforms see: "Integral Transforms and their Applications", Brian Davies, Springer-Verlag, 1978.

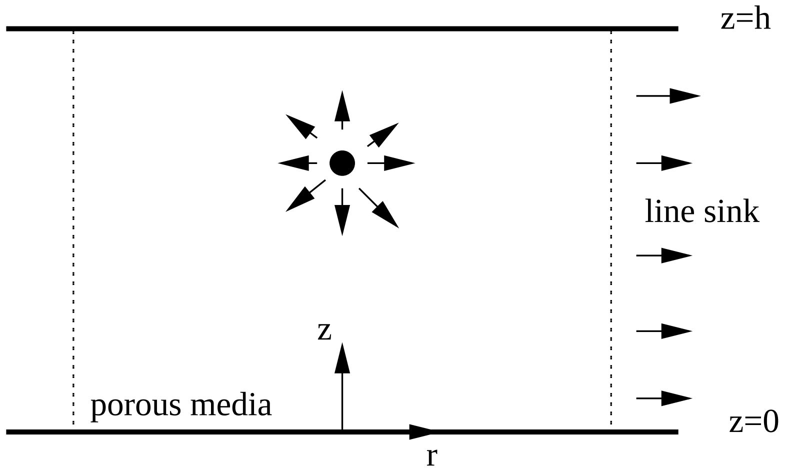

Figure 5.1: Source/sink configuration

5.2.2 Hankel Transform Example

Barry, Aldis and Mercer, "Injection of Fluid into a Layer of Deformable Porous Medium", Applied Mechanics Reviews, v48, n10, 1995, pg 722, consider fluid injection into a porous medium in the context of biological tissue. They note though that their solution is also relevant to subsurface fluid flow. The configuration they considered has a point source and a line sink as shown in Figure 5.1.

Barry et al. solve equations for both pressure and stress, however we will consider only the pressure equation. The pressure is governed by:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} = \alpha \left[\frac{\delta(r-\rho)}{r} - \frac{\delta(r)}{r}\delta(z-z_o) \right] \tag{5.16}$$

The Dirac delta terms are used to impose the rate boundary conditions at the source and sink. When this equation is Hankel transformed it becomes:

$$\frac{\partial^2 \bar{p}}{\partial z^2} - \lambda^2 \bar{p} = -\alpha(\delta(z-z_0) - J_0(\lambda p)) \tag{5.17}$$

where $\bar{p}$ is the Hankel transform of p;
The solution for $\bar{p}$ is:

$$\bar{p} = \alpha \frac{cosh(\lambda z_0)}{\lambda sinh(\lambda)} cosh(\lambda(z-1)) - \frac{J_0(\lambda\rho)}{\lambda^2}, \qquad z[z_0, 1] \tag{5.18}$$

$$\bar{p} = \alpha \frac{cosh(\lambda(z_0 - 1))}{\lambda sinh(\lambda)} cosh(\lambda(z)) - \frac{J_0(\lambda\rho)}{\lambda^2}, \qquad z[0, z_0] \tag{5.19}$$

These expressions were inverted by Barry et al. using a routine from the NAG library.

Chapter 6

Green's Functions

6.1 Theoretical Concepts

a.) adjoint operators
b.) Dirac delta function
c.) Green's function

"Green's Functions", G.F. Roach, Cambridge University Press, 1967
"Application of Green's Functions in Science and Engineering", Michael Greenberg, Prentice-Hall, 1971

6.1.1 Adjoint Operator

We will work in terms of a differential operator, L (note this is not the same as $\mathcal{L}$ for the Laplace transform). L operates on a function, u for example e.g.

$$L = \frac{d^2}{dx^2} + \frac{d}{dx} \tag{6.1}$$

$$Lu = \frac{d^2u}{dx^2} + \frac{du}{dx} \tag{6.2}$$

The adjoint of L is written L^*. It is defined by multiplying Lu by another (arbitrary) function v and integrating:

$$\int vLu \ dx = boundary \ \ terms + \int uL^*v \ dx \tag{6.3}$$

Example:

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x) \tag{6.4}$$

i.e.

$$Lu = a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u \tag{6.5}$$

What is L^*?

$$A = \int v[au'' + bu' + cu]dx \tag{6.6}$$

Integrate by parts (term by term):

$$\int vau''\,dx = vau' - \int u'(va)'\,dx = vau' - (va)'u + \int u(va)''\,dx \tag{6.7}$$

$$\int vbu'\,dx = uvb - \int u(vb)'\,dx \tag{6.8}$$

$$\Rightarrow A = vau' - (va)'u + uvb + \int u[(va)'' - (vb)' + vc]\,dx = \int uL^*v \;\, dx \tag{6.9}$$

Now consider L^*v a little more:

$$L^*v = (va)'' - (vb)' + vc = (v'a + a'v')' - v'b - b'v + vc \tag{6.10}$$

$$= v''a + a'v' + a''v + a'v' - v'b - b'v + vc = av'' + (2a' - b)v' + (a'' - b' + c)v \tag{6.11}$$

$$\Rightarrow L^* = a(x)\frac{d}{dx^2} + (2a'(x) - b(x))\frac{d}{dx} + (a''(x) - b'(x) + c(x)) \tag{6.12}$$

Self-Adjointness

If $L = L^*$ and the boundary terms vanish, the operator L is known as a self-adjoint operator. In the example above, L is self-adjoint if:

$$2a' - b = b \tag{6.13}$$

$$a'' - b' + c = c \tag{6.14}$$

$$\Rightarrow b = a' \tag{6.15}$$

i.e. if $b = a'$ then L is self adjoint. The importance of self adjoint operators will become clearer we discuss Green's functions.

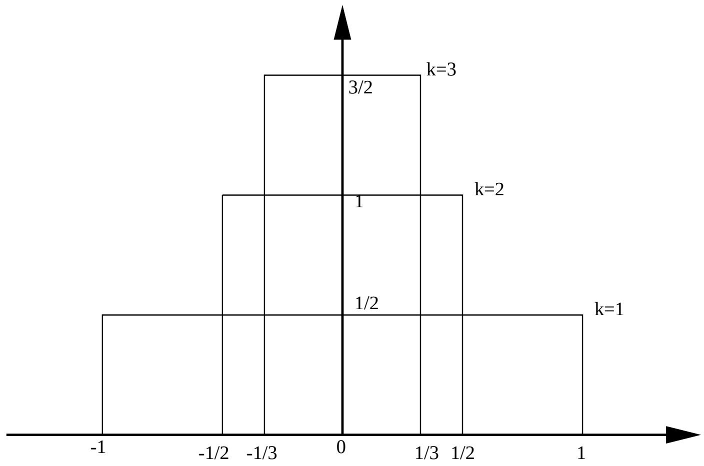

Figure 6.1: W_k function

6.1.2 The Dirac Delta Function

The Dirac delta function is written as $\delta(x)$ and is used to describe point source. Begin by considering a function, W_k:

$$W_k = \begin{array}{ll} \frac{k}{2}, & |x| < \frac{1}{k} \\ 0, & otherwise \end{array} \tag{6.16}$$

Now think about what happens as $k \to \infty$. The area under W_k is:

$$\lim_{k\to\infty} \int_{-\frac{1}{k}}^{\frac{1}{k}} \frac{k}{2}\ dx = \frac{k}{2} \int_{-\frac{1}{k}}^{\frac{1}{k}}\ dx \tag{6.17}$$

$$= \frac{k}{2}\left[\frac{1}{k} + \frac{1}{k}\right] = 1 \tag{6.18}$$

Define $\delta(x) = \lim_{k\to\infty} W_k$

$$\Rightarrow \int_{-\infty}^{\infty} \delta(x) = 1.0 \tag{6.19}$$

$$\delta(x) = \begin{array}{l} 0, x \neq 0 \\ \infty, x = 0 \end{array} \tag{6.20}$$

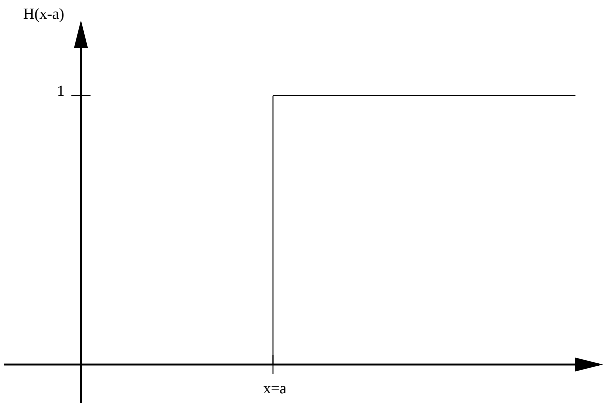

Figure 6.2: Heavisde function

Integrals Involving $\delta(x)$

$$\int_{-\infty}^{\infty} \delta(x)h(x)dx = h(0) \tag{6.21}$$

i.e. multiplying by $\delta(x)$ and integrating gives $h(0)$. Similarly:

$$\int_{-\infty}^{\infty} \delta(x-a)h(x)dx = h(a) \tag{6.22}$$

Derivatives Involving $\delta(x)$

This is easier to consider in integral form.

$$\int_{-\infty}^{\infty} \delta'(x)h(x)dx = \delta(x)h(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x)h'(x)\ dx \tag{6.23}$$

$$= - \int_{-\infty}^{\infty} \delta(x)h'(x)\ dx \tag{6.24}$$

More generally:

$$\int_{-\infty}^{\infty} \delta^n(x-a)h(x)dx = (-1)^n h^n(a) \tag{6.25}$$

Heaviside Step Functions and the Dirac Delta Function

The Heaviside step function is written $H(x-a)$.
Consider:

$$\int_{-\infty}^{\infty} H'(x-a)h(x)\ dx = H(x-a)h(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H(x-a)h'(x)\ dx \tag{6.26}$$

$$= h(\infty) - \int_{-\infty}^{\infty} H(x-a)h'(x)dx \tag{6.27}$$

$$= h(\infty) - \int_{a}^{\infty} h'(x)dx \tag{6.28}$$

$$= h(\infty) - (h(\infty) - h(a)) = h(a) \tag{6.29}$$

$$\int_{-\infty}^{\infty} \delta(x-a)h(x)\ dx \tag{6.30}$$

$$\Rightarrow H'(x-a) = \delta(x-a) \tag{6.31}$$

6.1.3 Green's Functions

Consider a differential equation written in operator form on a domain Ω:

$$Lu = f \tag{6.32}$$

where u is the unknown function and f is a forcing function. We will assume (initially) that L is self-adjoint, however this assumption will ultimately be relaxed. Using operator notation:

$$u = L^{-1}f \tag{6.33}$$

Since L is a differential operator we expect L^{-1} to be an integral operator. L^{-1} must satisfy the usual properties of an inverse:

$$LL^{-1} = L^{-1}L = I \tag{6.34}$$

Now define the inverse operator as:

$$L^{-1}f = \int_{\Omega} G(x, x_i)f(x_i)dx_i \tag{6.35}$$

To find the Green's function $G(x, x_i)$ for the problem consider what happens when $Lu = f$ is multiplied by G and integrated over the domain:

$$\int_{\Omega} GLu(x_i)dx_i = \int_{\Omega} uL^*Gdx_i = \int_{\Omega} Gf(x_i)dx_i \tag{6.36}$$

This equation shows that (6.35) is an appropriate definition of L^{-1} if:

$$L^*G(x, x_i) = LG(x, xi) = \delta(x_i - x) \tag{6.37}$$

The boundary conditions for this problem can be found by setting the boundary terms to zero.

$$u = \int_{\Omega} G(x, x_i)f(x_i)dx_i \tag{6.38}$$

This derivation assumes L is a self adjoint operator, we will return to the non-self adjoint case in another section.

6.2 Green's Function Examples

6.2.1 Self Adjoint Problem

Use Green's functions to solve:

$$u''(x) = \phi(x), \qquad x[0,1] \tag{6.39}$$

$$u(0) = u(1) = 0 \tag{6.40}$$

Step 1: Find L^*

Note we are working with $u(x_i)$ not $u(x)$.

$$\int_0^1 Gu''dx_i = Gu'|_0^1 - \int_0^1 u'G'dx_i \tag{6.41}$$

$$= Gu'|_0^1 - uG'|_0^1 + \int_0^1 uG''dx_i \tag{6.42}$$

i.e.

$$L^* = \frac{d^2}{dx^2} \tag{6.43}$$

Step 2: Consider Boundary Terms

To ensure the problem is self-adjoint we will zero the boundary terms by imposing boundary conditions on G:

$$G(x,1)u'(1) - G(x,0)u'(0) = 0 \tag{6.44}$$

$$G'(x,1)u(1) - G'(x,0)u(0) = 0 \tag{6.45}$$

Since $u(1) = u(0) = 0$ we require:

$$G(x,1) = 0 \tag{6.46}$$

$$G(x,0) = 0 \tag{6.47}$$

Step 3: Solve for G

$$\frac{d^2G}{dx_i^2} = \delta(x_i - x) \tag{6.48}$$

$$\Rightarrow \frac{dG}{dx_i} = H(x_i - x) + Ax_i \tag{6.49}$$

$$\Rightarrow G = (x_i - x)H(x_i - x) + Ax_i + B \tag{6.50}$$

Now use boundary conditions to solve for A and B:

$$G(x, 0) = 0 \tag{6.51}$$

$$\Rightarrow B = -xH(-x) = 0 \tag{6.52}$$

$$G(x, 1) = 0 \tag{6.53}$$

$$\Rightarrow (1 - x)H(1 - x) + A = 0 \tag{6.54}$$

$$\Rightarrow A = -(1 - x) \tag{6.55}$$

Substitute A and B back into G:

$$G = (x_i - x)H(x_i - x) + (x - 1)x_i \tag{6.56}$$

Step 4: Solve for u

$$u(x) = \int_0^1 [(x_i - x)H(x_i - x) + (x - 1)x_i]\phi(x_i)dx_i \tag{6.57}$$

Symmetry and Interpretation of G

Note the symmetry in G.

When $x_i < x$:

$$G = (x - 1)x_i \tag{6.58}$$

When $x_i > x$:

$$G = (x_i - 1)x \tag{6.59}$$

This symmetry is something that should be expected for self adjoint problems. Physically G can be interpreted as the deflection of a beam in response to an incremental load $\phi(x_i)dx_i$ at point x_i.

6.2.2 Non-Self Adjoint Problem

Use Green's functions to solve:

$$u''(x) + 3u'(x) + 2u = f, \qquad x[0,1] \tag{6.60}$$

$$u(1) = 2u(0) \tag{6.61}$$

$$u'(1) = a \tag{6.62}$$

Step 1: Find L^*

$$\int_0^1 GLudx_i = Gu'|_0^1 - \int_0^1 u'G'dx_i + 3Gu|_0^1 - 3\int_0^1 uG'dx_i + 2\int_0^1 uGdx_i \tag{6.63}$$

$$= Gu'|_0^1 - G'u|_0^1 + \int_0^1 uG''dx_i + 3Gu|_0^1 - 3\int_0^1 uG'dx_i + 2\int_0^1 uGdx_i \tag{6.64}$$

$$\Rightarrow L^* = \frac{d^2}{dx^2} - 3\frac{d}{dx} + 2 \tag{6.65}$$

The problem is not self adjoint.

Step 2: Consider Boundary Terms

$$G(x,1)u'(1) - G'(x,1)u(1) + 3G(x,1)u(1)$$
$$-G(x,0)u'(0) + G'(x,0)u(0) - 3G(x,0)u(0) = 0 \tag{6.66}$$

Substitute in the boundary conditions for u:

$$aG(x,1) + u(0)[-2G'(x,1) + 6G(x,1) + G'(x,0)] = 0 \tag{6.67}$$

$$G(x,0) = 0 \tag{6.68}$$

We can only specify two boundary conditions on G. The mixed boundary condition in this problem means that this is not enough to zero all the boudnary terms. We will choose to carry $aG(x,1)$ through the problem and set:

$$-2G'(x,1) + 6G(x,1) + G'(0,x) = 0 \tag{6.69}$$

As in the self adjoint problem we will multiply Lu by G and integrate over Ω.

$$\int_0^1 GLudx_i = boundary\ terms + \int_0^1 uL^*Gdx_i = \int_0^1 Gfdx_i \tag{6.70}$$

By zeroing out as many boundary terms as possible and choosing $L^*G = \delta(x_i - x)$ this becomes:

$$\int_0^1 Gf dx_i = aG(x,1) + u(x) \tag{6.71}$$

Step 3: Solve for G

It is convenient to solve the problem in two parts.

$$\frac{d^2G}{dx_i^2} - 3\frac{dG}{dx_i} + 2G = 0, \qquad 0 \leq x_i < x \tag{6.72}$$

$$\frac{d^2G}{dx_i^2} - 3\frac{dG}{dx_i} + 2G = 0, \qquad x < x_i \leq 1 \tag{6.73}$$

The singularity at $x_i = x$ will be handled by imposing conditions on the constants of integration. The general solution to this problem is:

$$G = Ae^{x_i} + Be^{2x_i}, \qquad 0 \leq x_i < x \tag{6.74}$$

$$G = Ce^{x_i} + De^{2x_i}, \qquad x < x_i \leq 1 \tag{6.75}$$

The constants A, B, C and D can be solved for using the boundary conditions and some additional conditions to handle the singularity:

$$G(x,0) = 0 \tag{6.76}$$

$$\Rightarrow A + B = 0 \tag{6.77}$$

$$-2G'(x,1) + 6G(x,1) + G'(0,x) = 0 \tag{6.78}$$

$$\Rightarrow -2(Ce + 2De^2) + 6(Ce + De^2) + (A + 2B) = 0 \tag{6.79}$$

$$A + 2B + 4Ce + 2De^2 = 0 \tag{6.80}$$

We will require that G is continuous at $x_i = x$:

$$\Rightarrow Ae^x + Be^{2x} = Ce^x + De^{2x} \tag{6.81}$$

Finally we will consider what happens when we integrate past $x_i = x$:

$$\int_{x-0}^{x+0} G'' - 3G' + 2G dx_i = \int_{x-0}^{x+0} \delta(x_i - x) dx_i \tag{6.82}$$

$$\Rightarrow \frac{dG}{dx_i}|_{x-0}^{x+0} - 3G|_{x-0}^{x+0} + 2\int_{x-0}^{x+0} G dx_i = 1 \tag{6.83}$$

Because we have required G to be continuous the second and third terms in this equation are zero so we have:

$$(Ce^x + 2De^{2x}) - (Ae^x + 2Be^{2x}) = 1 \tag{6.84}$$

We now have four equations to solve for the constants. The final solution for G is:

$$G = \frac{1}{k}(2e^{2(1-x)} - 4e^{1-x})(e^{x_i} - e^{2x_i}), \quad 0 \le x_i \le x \tag{6.85}$$

$$G = \frac{1}{k}(2e^{2-x} - 2e^2 - 1)e^{x_i - x} + (4e - 4e^{1-x} + e^{-x})e^{2x_i - x}, \quad x \le x_i <= 1 \tag{6.86}$$

where

$$k = 1 - 4e + 2e^2 \tag{6.87}$$

Step 4: Solve for u

$$u(x) = -aG(x, 1) + \int_0^1 G(x, x_i) f(x_i) dx_i \tag{6.88}$$

6.3 Partial Differential Equations

Consider a general second order partial differential equation, on a domain Ω:

$$Lu = Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = f \tag{6.89}$$

This assumes the independent variables are x and y but x are t are also possible. This equation can be classified according to A,B and C.

$B^2 - AC < 0$, elliptic
$B^2 - AC = 0$, parabolic
$B^2 - AC > 0$, hyperbolic

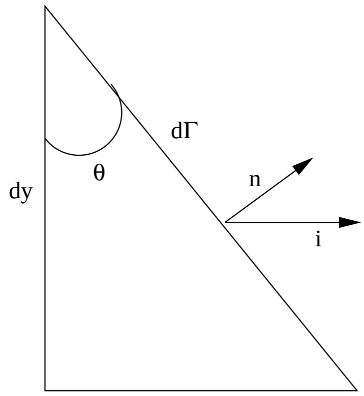

Figure 6.3: Relationship of dy to $d\Gamma$

6.3.1 Adjoint Operator

As before the adjoint operator is defined in terms of the following integral:

$$\iint_\Omega vLu\, d\Omega = boundary\ \ terms + \int\int uL^* v\, d\Omega \tag{6.90}$$

We will work out the first term in the general case for the adjoint operator:

$$\int\int vAu_{xx}d\Omega = \int_{y_1}^{y_2}\left\{\int_{x_1}^{x^2} vAu_{xx}dx\right\}dy \tag{6.91}$$

$$= \int_{y_1}^{y_2}\left\{vAu_x|_{x_1}^{x_2} - \int_{x_1}^{x_2}(vA)_x u_x dx\right\}dy \tag{6.92}$$

$$= \int_{y_1}^{y_2}\left\{[vAu_x - (vA)_x u]_{x_1}^{x_2} + \int_{x_1}^{x_2}(vA)_{xx}udx\right\}dy \tag{6.93}$$

$$= \int_{y_1}^{y_2}[vAu_x - (vA)_x u]_{x_1}^{x_2}dy + \iint (vA)_{xx}ud\Omega \tag{6.94}$$

Now consider the boundary integration more carefully. The boundary (Γ) is an arbitrary function of x and y.

$$dy = d\Gamma\cos\theta = \vec{i}\cdot\vec{n}d\Gamma \tag{6.95}$$

Therefore the integral being evaluated is:

$$\int_\Gamma [vAu_x - (vA)_x u]\vec{i}\cdot\vec{n}d\Gamma + \iint_\Omega (vA)_{xx}ud\Omega \tag{6.96}$$

Treating all terms in this manner gives the following relationship between L nand L^*:

$$\iint vLu \, d\Omega = \int_\Gamma (M\vec{i} + N\vec{j}) \cdot \vec{n} d\Gamma + \iint_\Omega uL^* v \, d\Omega \tag{6.97}$$

where

$$L^* v = (Av)_{xx} + 2(Bv)_{xy} + (Cv)_{yy} - (Dv)_x - (Ev)_y + Fv \tag{6.98}$$

$$M = Avu_x - u(Av)_x + 2vBu_y + Duv \tag{6.99}$$

$$N = -2u(Bv)_x + Cvu_y - u(Cv)_y + Euv \tag{6.100}$$

Common Operators

Equation	Lu	L^*v
Laplace	$\nabla^2 u$	$\nabla^2 v$
Helmholtz	$\nabla^2 u + k^2 u$	$\nabla^2 v + k^2 v$
Diffusion	$\kappa u_t - u_{xx}$	$-\kappa v_t - v_{xx}$
Wave	$c^2 u_{xx} - u_{tt}$	$c^2 v_{xx} - v_{tt}$

Of these four the diffusion equation is the only one that is not self-adjoint.

It can be proven that:

$$A_x + B_y = D \tag{6.101}$$

and

$$B_x + C_y = E \tag{6.102}$$

are necessary and sufficient conditions for $L = L^*$.

6.3.2 The Delta Function is Two Dimensions

Like $\delta(x)$, $\delta(x, y)$ can be seen as the limit of a sequence of other functions.

$$\delta(x - x_i, y - y_i) = lim_{k\to\infty} W_k(r) \tag{6.103}$$

where

$$r = \sqrt{(x - x_i)^2 + (y - y_i)^2} \tag{6.104}$$

and

$$W_k = \frac{k^2}{\pi}, \quad r \leq \frac{1}{k} \tag{6.105}$$

$$= 0, \quad r > \frac{1}{k} \tag{6.106}$$

and/or

$$W_k(r) = \frac{ke^{-kr^2}}{\pi} \tag{6.107}$$

The two dimensional delta function has similar properties to the one dimensional delta function:

$$\iint_{d\Omega} \delta(x - x_i, y - y_i)h(x, y)d\Omega = h(x_i, y_i) \tag{6.108}$$

$$\delta(x - x_i, y - y_i) = \delta(x - x_i)\delta(y - y_i) \tag{6.109}$$

6.3.3 Constructing Green's Functions

When working in two dimensions is usually easier to construct the Green's function for a given problem as the sum of two Green's functions:

$$G = G_f + G_b \tag{6.110}$$

where G_f satisifes $L^*G_f = \delta(x_i - x, y_i - y)$ **without** taking account of any particular boundary conditions on G or G_f. The function G_f is known as a free-space Green's function. The function G_b takes account of the boundary terms for a particular problem.

Example: Two-Dimensional Laplace Equation

$$L^*G_f = \nabla_f^G = \delta(x_i - x, y_i - y) \tag{6.111}$$

As before we will begin by solving the problem away from the singularity where the delta function is zero, then will consider the singularity separately.

$$\nabla^2 G_f = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial G_f}{\partial r}) = 0 \tag{6.112}$$

$$\Rightarrow r\frac{\partial G_f}{\partial r} = A \tag{6.113}$$

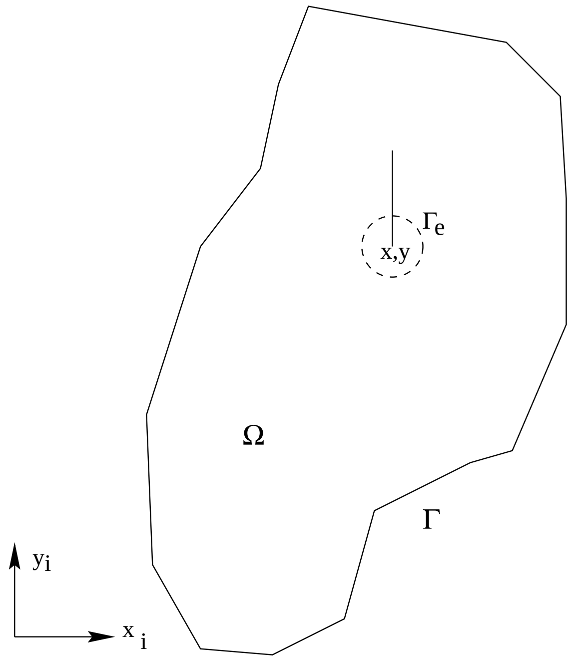

Figure 6.4: Integration around singularity

$$\Rightarrow \frac{\partial G_f}{\partial r} = \frac{A}{r} \tag{6.114}$$

$$\Rightarrow \frac{\partial G_f}{\partial r} = \frac{A}{r} \tag{6.115}$$

$$\Rightarrow G_f = Alnr + B \tag{6.116}$$

where

$$r = \sqrt{(x_i - x)^2 + (y_i - y)^2} \tag{6.117}$$

Now consider the singularity by integrating over a disc surrounding the singularity as shown in Figure 6.4.

$$\iint_{\Omega_e} \nabla^2 G_f d\Omega = \iint_{\Omega_e} \delta(x_i - x, y_i - y) d\Omega \tag{6.118}$$

Use Green's second identity to convert the domain integral on the left to a boundary integral. In general for two function f and g Green proved:

$$\iint_{\Omega} (f\nabla^2 g - g\nabla^2 f) d\Omega = \int_{\Gamma} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) d\Gamma \tag{6.119}$$

$$\Rightarrow \iint_{\Omega_e} \nabla^2 G_f d\Omega = \int_{\Gamma_e} \frac{\partial G_f}{\partial r} dr \tag{6.120}$$

$$= \int_0^{2\pi} \frac{\partial G_f}{\partial r} r d\theta \tag{6.121}$$

Now substitute $G_f = Alnr + B$:

$$\Rightarrow = A \int_0^{2\pi} \frac{1}{r} r d\theta \tag{6.122}$$

Recalling the left hand side of Equation 6.118 must equal 1 we have:

$$2\pi A = 1 \tag{6.123}$$

i.e.

$$A = 1 \tag{6.124}$$

Since we are not considering any boundary conditions we can not solve for B explicity so we will set it (arbitrarily) to zero, i.e.:

$$G_f = \frac{1}{2\pi} lnr \tag{6.125}$$

Example: One-Dimensional Diffusion Equation

The diffusion equation is:

$$Lu = \kappa \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \tag{6.126}$$

Referring to the general expression for L and L^* we have the following L^*:

$$L^* = -\kappa \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \tag{6.127}$$

To find the (free-space) Green's function we will solve:

$$L^* G_f = \kappa \frac{\partial G_f}{\partial \tau} + \frac{\partial^2 G_f}{\partial x_i^2} = -\delta(x_i - x)\delta(\tau - t) \tag{6.128}$$

We'll use a Fourier transform to do this, specifically:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x_i) e^{-ix_i s} dx_i \tag{6.129}$$

$$\Rightarrow \kappa \frac{\partial \hat{G}_f}{\partial \tau} - s^2 \hat{G}_f = -\delta(\tau - t) \frac{1}{\sqrt{2\pi}} e^{-ixs} \tag{6.130}$$

As before consider solving two problems, one on each side of the singularity (where the delta function is zero):

$$\kappa \frac{\partial \hat{G}_f}{\partial \tau} - s^2 \hat{G}_f = 0, \quad \tau > t \tag{6.131}$$

$$\kappa \frac{\partial \hat{G}_f}{\partial \tau} - s^2 \hat{G}_f = 0, \quad \tau < t \tag{6.132}$$

This equation can be readily solved for $\hat{G}_f$:

$$\hat{G}_f = A e^{s^2 \tau} \kappa, \quad \tau > t \tag{6.133}$$

$$\hat{G}_f = B e^{s^2 \tau} \kappa, \quad \tau < t \tag{6.134}$$

Now take account of the singularity by integrating past it:

$$\kappa \int_{t-0}^{t+0} \frac{d\hat{G}_f}{d\tau} d\tau - s^2 \int_{t-0}^{t+0} \hat{G}_f d\tau = -\frac{1}{\sqrt{2\pi}} \int_{t-0}^{t+0} \delta(\tau - t) e^{-ixs} d\tau \tag{6.135}$$

We require $\hat{G}_f$ to be continuous so the second integral in the above equation vanishes to give:

$$\kappa (A e^{s^2 \tau} \kappa - B e^{s^2 \tau} \kappa) = -\frac{1}{\sqrt{2\pi}} e^{-ixs} \tag{6.136}$$

To solve for A and B we need one more equation. This time we will consider a physical argument. The Green's function represent the response of a system to a unit input applied at time τ and location x_i. So before time τ we do not expect change in the system i.e G will be zero for $t < \tau$. Therefore the constant A is zero.
Now solve for B:

$$B = \frac{1}{\kappa \sqrt{2\pi}} e^{-ixs} e^{\frac{-s^2 t}{\kappa}} \tag{6.137}$$

$$\hat{G}_f = 0, \quad \tau > t \tag{6.138}$$

$$\hat{G}_f = \frac{1}{\kappa \sqrt{2\pi}} e^{-ixs - \frac{s^2 t}{\kappa} + \frac{s^2 \tau}{\kappa}} \quad \tau > t \tag{6.139}$$

$\hat{G}_f$ can be written more compactly as:

$$\hat{G}_f = \frac{H(t-\tau)}{\kappa\sqrt{2\pi}} e^{-ixs - \frac{s^2 t}{\kappa} + \frac{s^2 \tau}{\kappa}} \tag{6.140}$$

The Fourier transform can be inverted to give:

$$G_f = \frac{H(t-\tau)}{\sqrt{4\pi\kappa(t-\tau)}} e^{\frac{-\kappa(x_i - x)^2}{4(t-\tau)}} \tag{6.141}$$

6.4 The Newman Product Theorem

This theorem is not specific to the solution of the differential equations that give rise to Green's functions ($L^* G = \delta(x_i - x)$ etc). The theorem allows us to find solutions to differential equations in multiple dimensions from solutions to one-dimensional problems. We will consider the diffusion equation in three dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\kappa}\frac{\partial u}{\partial t} \tag{6.142}$$

on the domain

$$a_1 \le x \le b_1 \tag{6.143}$$

$$a_2 \le y \le b_2 \tag{6.144}$$

$$a_3 \le z \le b_3 \tag{6.145}$$

with initial conditions

$$u(x, y, z, 0) = U(x)V(y)W(z) \tag{6.146}$$

Note being able to express the initial condition in this "product" form is essential.

The boundary conditions can be constant u, constant flux or mixed on each edge.

Now consider solving three one-dimensional problems:

$$\frac{\partial^2 u_1}{\partial x^2} = \frac{1}{\kappa}\frac{\partial u_1}{\partial t}, \qquad a_1 \le x \le b_1 \tag{6.147}$$

$$\frac{\partial^2 u_2}{\partial y^2} = \frac{1}{\kappa}\frac{\partial u_2}{\partial t}, \qquad a_2 \leq y \leq b_2 \tag{6.148}$$

$$\frac{\partial^2 u_3}{\partial z^2} = \frac{1}{\kappa}\frac{\partial u_3}{\partial t}, \qquad a_3 \leq y \leq b_3 \tag{6.149}$$

The initial conditions for this set of equations are:

$$u_1(x,0) = U(x) \tag{6.150}$$

$$u_2(y,0) = V(y) \tag{6.151}$$

$$u_3(z,0) = W(z) \tag{6.152}$$

and the boundary conditions are taken from the original problem. The solution for u is:

$$u(x,y,x,t) = u_1(x,t)u_2(y,t)u_3(z,t) \tag{6.153}$$

Proof: Substitute (6.153) into the diffusion equation (6.142):

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u_1 u_2 u_3}{\partial x^2} + \frac{\partial^2 u_1 u_2 u_3}{\partial y^2} + \frac{\partial^2 u_1 u_2 u_3}{\partial z^2} \tag{6.154}$$

Expanding the derivatives gives:

$$= u_2 u_3 \frac{\partial^2 u_1}{\partial x^2} + u_1 u_3 \frac{\partial^2 u_2}{\partial y^2} + u_1 u_2 \frac{\partial^2 u_3}{\partial z^2} \tag{6.155}$$

Now substitute in (6.147):

$$= u_2 u_3 \frac{1}{\kappa}\frac{\partial u_1}{\partial t} + u_1 u_3 \frac{1}{\kappa}\frac{\partial u_2}{\partial t} + u_1 u_2 \frac{1}{\kappa}\frac{\partial u_3}{\partial t} \tag{6.156}$$

$$= \frac{1}{\kappa}\frac{\partial u}{\partial t} \tag{6.157}$$

For an example involving radial coordinates see Carslaw and Jaegar, page 33.

6.5 Gringarten and Ramey

Gringarten and Ramey, "The Use of Source and Green's Functions in Solving Unsteady-Flow Problems in Reservoirs". SPEJ, October 1973, pg 285, applied Green's functions and the Newman product theorem to reservoir engineering problems. Gringarten and Ramey derived both free-space Green's functions and Green's functions satisfying boundary conditions for rectangular, homogeneous, anisotropic reservoirs. Gringarten and Ramey define both Green's functions and "source" functions. Source functions act over a region while Green's functions act a given point in one, two or three dimensions. Source functions can be determined by integrating Green's functions over an appropriate region (corresponding to a well or fracture).

The basic result that Gringarten and Ramey use is the Green functions for a point source in a one dimensional reservoir:

$$G = \frac{1}{\sqrt{4\pi\eta t}} e^{-\frac{(x-x_w)^2}{4\eta t}} \tag{6.158}$$

where

$$\eta \nabla^2 p - \frac{\partial p}{\partial t} = 0 \tag{6.159}$$

Note that this one dimensional source corresponds to an infinite planar source in three dimensions (see Figure 6.5). This source can be integrated over a region of width x_f to produce a slab source as shown in Figure 6.6:

$$S = \frac{1}{2}\left[erf\left(\frac{x-(x_w-x_f/2)}{\sqrt{4\eta t}}\right) - erf\left(\frac{x-(x_w+x_f/2)}{\sqrt{4\eta t}}\right)\right] \tag{6.160}$$

The product of the inifinite plan source and the infinite line source gives the point source solution (see Figure 6.7).

The set of free-space solutions Gringarten and Ramey generated is given by I(x) to VI in Table 1 of their paper.

6.5.1 Adding Boundaries

Gringarten and Ramey added boundary conditions to the solutions for infinite plane and infinte slab sources by using the method of images (which we discussed at the start of the course. These solutions are VII(x) to XII(x) in Table 2 of their paper. For instance consider constant pressure boundaries

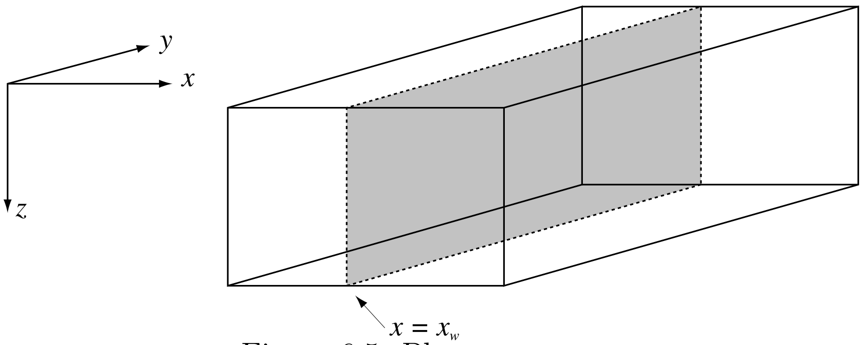

Figure 6.5: Planar source

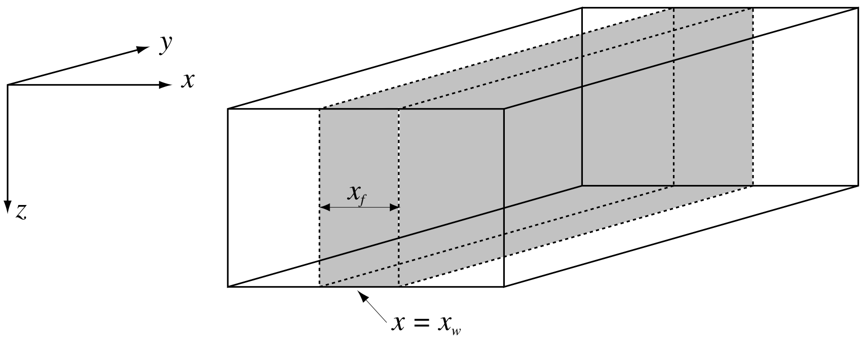

Figure 6.6: Slab source

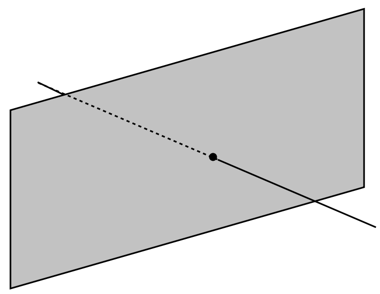

Figure 6.7: 3D Point source as a product of a line source and a plane source

at $x = 0$ and $x = x_e$. To ensure the pressure in the final solution is zero we will require that $G = 0$ on these boundaries (since G represent the effect of a unit flow rate at the well at any location in space). The sequence of images is shown in Figure 6.8.

The sequence extends infinitely in both directions. The Green's function that corresponds with this sequence is:

$$G = \frac{1}{\sqrt{4\pi\eta t}} \left\{ e^{-\frac{(x-x_w)^2}{4\eta t}} - e^{-\frac{(x+x_w)^2}{4\eta t}} - e^{-\frac{(x-2x_e+x_w)^2}{4\eta t}} + e^{-\frac{(x+2x_e-x_w)^2}{4\eta t}} + ... \right\} \tag{6.161}$$

$$G = \frac{1}{\sqrt{4\pi\eta t}} \left\{ \sum_{n=-\infty}^{\infty} e^{-\frac{(x-x_w-2nx_e)^2}{4\eta t}} - e^{-\frac{(x+x_w-2nx_e)^2}{4\eta t}} \right\} \tag{6.162}$$

This series can have convergence problems. It is better to expand it using Poisson's summation formula, which is:

$$\sum_{-\infty}^{\infty} f(\alpha n) = \frac{\sqrt{2\pi}}{\alpha} \sum_{n=-\infty}^{\infty} F\left(\frac{2\pi n}{\alpha}\right) \tag{6.163}$$

where F is the Fourier transform of f.

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixs} dx \tag{6.164}$$

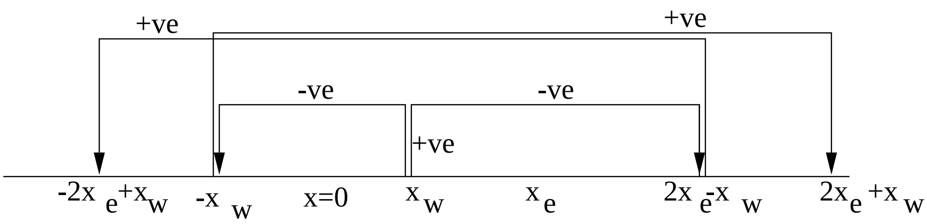

Figure 6.8: Sequence of image wells

We will need

$$F\left(e^{-\frac{(x-x_w-2nx_e)^2}{4\eta t}}\right) \tag{6.165}$$

and

$$F\left(e^{-\frac{(x+x_w-2nx_e)^2}{4\eta t}}\right) \tag{6.166}$$

Begin by defining a new variable, $\tilde{x} = 2nx_e$. We will do the Fourier transform in terms of this variable. The first shift theorem can be applied to give:

$$F\left(e^{-\frac{(x-x_w-2nx_e)^2}{4\eta t}}\right) = e^{-is(x-xw)}F\left(e^{-\frac{-(2nx_e)^2}{4\eta t}}\right) \tag{6.167}$$

$$= e^{-is(x-xw)}F\left(e^{-\frac{\tilde{x}^2}{4\eta t}}\right) \tag{6.168}$$

$$= e^{-is(x-xw)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-is\tilde{x}}e^{-\frac{\tilde{x}^2}{4\eta t}}d\tilde{x} \tag{6.169}$$

We will evaluate this integral by completing the square in the exponent. The exponent is:

$$-\left(\frac{\tilde{x}^2}{4\eta t} + is\tilde{x}\right) = \left(\left(\frac{\tilde{x}}{\sqrt{4\eta t}} + \frac{\sqrt{4\eta t}is}{2}\right)^2 + \eta ts^2\right) \tag{6.170}$$

Recall the following useful integral which will help us evaluate the integral we require for the Fourier transform:

$$\int_{-\infty}^{\infty} e^{\frac{-z^2}{A^2}}dz = \sqrt{\pi}A \tag{6.171}$$

Using this result the Fourier transform we need is:

$$e^{-is(x-x_w)}\frac{1}{\sqrt{2\pi}}\sqrt{\pi}\sqrt{4\pi\eta t}e^{-s^2\eta t} = \sqrt{2\eta t}e^{-is(x-x_w)}e^{-s^2\eta t} \tag{6.172}$$

Now return to the summation:

$$\sum_{n=-\infty}^{\infty} e^{-\frac{(x-x_w-2nx_e)^2}{4\eta t}} = \sum_{n=-\infty}^{\infty} \frac{\sqrt{2\pi}}{2x_e}\sqrt{2\eta t}e^{-is(x-x_w)}e^{-s^2\eta t} \tag{6.173}$$

where we have replace the α in Poission's summation formula by $2x_e$. We still need to consider the s.

$$s = \frac{2\pi n}{\alpha} = \frac{2\pi n}{2x_e} = \frac{\pi n}{x_e} \tag{6.174}$$

So the summation for the Green's function is:

$$G = \frac{1}{\sqrt{4\pi\eta t}}\left\{\sum_{n=-\infty}^{\infty} e^{-\frac{(x-x_w-2nx_e)^2}{4\eta t}} - e^{-\frac{(x+x_w-2nx_e)^2}{4\eta t}}\right\} \tag{6.175}$$

$$= \frac{1}{\sqrt{4\pi\eta t}}\frac{\sqrt{4\pi\eta t}}{2x_e}\left\{\sum_{n=-\infty}^{\infty} e^{-\frac{i\pi n(x-x_w)}{x_e}}e^{-\frac{\pi^2 n^2\eta t}{x_e^2}} - e^{-\frac{i\pi n(x+x_w)}{x_e}}e^{-\frac{\pi^2 n^2\eta t}{x_e^2}}\right\} \tag{6.176}$$

$$= \frac{1}{2x_e}\left\{\sum_{n=-\infty}^{\infty} e^{-\frac{-\pi^2 n^2\eta t}{x_e^2}}\left[e^{-\frac{i\pi n(x-x_w)}{x_e}} - e^{-\frac{i\pi n(x+x_w)}{x_e}}\right]\right\} \tag{6.177}$$

The terms in square brackets can be expanded in terms of sines and cosines to give:

$$\left[e^{-\frac{i\pi n(x-x_w)}{x_e}} - e^{-\frac{i\pi n(x+x_w)}{x_e}}\right] = \left(cos(\frac{\pi nx}{x_e}) - isin(\frac{\pi nx}{x_e})\right)2isin(\frac{\pi nx_w}{x_e}) \tag{6.178}$$

The product of the cosine and sine terms is zero when summed (recall the integral of an even function times an odd function is also zero), so only the sine terms remain. Finally when we substitute back into G we have:

$$G = \frac{1}{x_e}\left\{\sum_{n=-\infty}^{\infty} e^{-\frac{\pi^2 n^2\eta t}{x_e^2}} sin(\frac{\pi nx}{x_e})sin(\frac{\pi nx_w}{x_e})\right\} \tag{6.179}$$

Because of the symmetry in the sine terms we have

$$G = \frac{2}{x_e}\left\{\sum_{n=1}^{\infty} e^{-\frac{\pi^2 n^2\eta t}{x_e^2}} sin(\frac{\pi nx}{x_e})sin(\frac{\pi nx_w}{x_e})\right\} \tag{6.180}$$

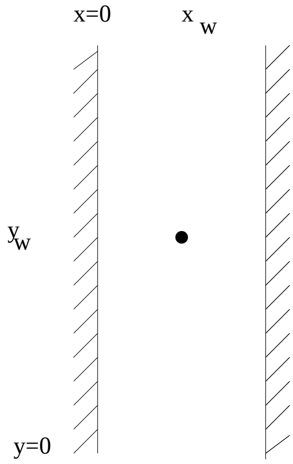

Figure 6.9: Well/boundary geometry

6.5.2 General Rectangular Reservoirs

The appropriate source functions for bounded rectangular reservoirs can be generated by applying Newmans's product theorem and using suitable one-dimensional solutions e.g. consider a fully completed well in a reservoir that is infinite in the y direction and has constant flux boundaries in x.

The appropriate source function is $I(y).VII(x)$

6.5.3 Recovering the Pressure

The pressure equation is:

$$\frac{k}{\mu}\nabla^2 p - \phi c \frac{\partial p}{\partial t} = q \tag{6.181}$$

where q represent sources and sinks caused by wells. This can be rearranged to give:

$$\eta \nabla^2 p - \frac{\partial p}{\partial t} = \frac{1}{\phi c} q \tag{6.182}$$

Once the appropriate source function has been determined the drawdown can be found by evaluating the following integral:

$$\Delta p(t) = \frac{1}{\phi c}\int_0^t q(\tau)S(t-\tau)d\tau \tag{6.183}$$

6.6 Green's Function Summary

- Trying to solve a differential equation of the form $Lu(x,t) = f(x,t)$ where u is an unknown function and f is a known forcing function.
- The first step is to determine the adjoint operator, L^*:

$$\iint_\Omega vLud\Omega = boundary\ \ terms + \int uL^*vd\Omega \tag{6.184}$$

- Determine specific boundary conditions for G by zeroing the boundary terms arising from L^*. If you only want a free-space Green's function don't worry about the boundary conditions.
- Solve for G: Multiply $Lu = f$ by G and integrate. Work in terms of dummy variables x_i and τ.

$$\iint_\Omega G(x,x_i,t,\tau)Lu(x_i,\tau)d\Omega = \iint_\Omega Gf(x_i,\tau)d\Omega \tag{6.185}$$

Now refer back to the adjoint equation:

$$\iint_\Omega G(x,x_i,t,\tau)Lu(x_i,\tau)d\Omega = boundary\ \ terms + \iint_\Omega u(x_i,\tau)L^*Gd\Omega \tag{6.186}$$

This equation implies that setting:

$$L^*G = \delta(x_i - x, \tau - t) \tag{6.187}$$

will allow us to solve for $u(x,t)$

$$u(x,t) = \iint_\Omega Gf(x_i,\tau)d\Omega - boundary\ \ terms \tag{6.188}$$

- Where's the singularity?
While we are solving for G the domain of the problem is x_i, τ so the singularity is at a point x, t, but ...
... once we have G and are thinking about what it means physically G is the effect on u (in the domain x,t) of a singularity at x_i, τ.

Chapter 7

Numerical Methods

7.1 Boundary Element Method

The boundary element method (BEM) is a numerical method which solves a differential equation ($Lu = f$) in an integral form. We'll begin by considering the Laplace equation, however other equations (including transient equations) ca be considered:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 \tag{7.1}$$

7.1.1 Derivation of the Boundary Integral Equation

The derivation of BEM begins from Green's second identity. For two arbitrary functions f and g Green proved the following:

$$\int\int_{\Omega} (f\nabla^2 g - g\nabla^2 f) d\Omega = \int_{\Gamma} f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} d\Gamma \tag{7.2}$$

where n is the normal to the boundary.

In our case we will take:

$$f = p \tag{7.3}$$

$$g = G = \frac{1}{2\pi} ln(1/r) \tag{7.4}$$

where

$$r = \sqrt{(x_i - x)^2 + (y_i - y)^2} \tag{7.5}$$

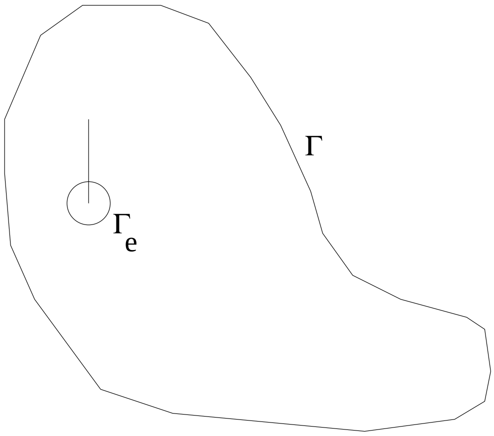

Figure 7.1: Location of singularity

and G is the free-space Green's function for the problem.

$$\Rightarrow \iint_{\Omega} (p\nabla^2 G - G\nabla^2 p) d\Omega = \int_{\Gamma} p\frac{\partial G}{\partial n} - G\frac{\partial p}{\partial n} d\Gamma \tag{7.6}$$

This equation is key to the derivation of BEM. It shows how we can convert domain integrals into boundary integrals for the same problem. To proceed further we'll break up the domain Ω into the sume of a circle surrounding the singularity (Ω_e) and the remainder, $\Omega - \Omega_e$. First consider the remainder:

$$\Rightarrow \iint_{\Omega-\Omega_e} (p\nabla^2 G - G\nabla^2 p) d\Omega = \int_{\Gamma+\Gamma_e} p\frac{\partial G}{\partial n} - G\frac{\partial p}{\partial n} d\Gamma \tag{7.7}$$

Within this domain both $\nabla^2 G = 0$ and $\nabla^2 p = 0$.

$$\Rightarrow 0 = \int_{\Gamma} p\frac{\partial G}{\partial n} - G\frac{\partial p}{\partial n} d\Gamma + \int_{\Gamma_e} p\frac{\partial G}{\partial n} - G\frac{\partial p}{\partial n} d\Gamma \tag{7.8}$$

Now consider the integral over Γ_e further, noting that $d\Gamma = rd\theta$:

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial r}\frac{\partial r}{\partial n} = \frac{1}{2\pi}\frac{-1}{r}(-1) = \frac{1}{2\pi r} \tag{7.9}$$

$$\int_{\Gamma_e} p\frac{\partial G}{\partial n} - G\frac{\partial p}{\partial n} d\Gamma = \frac{1}{2\pi}\int_0^{2\pi} [p\frac{1}{r} - ln\frac{1}{r}\frac{\partial p}{\partial n}] r d\theta \tag{7.10}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (p + rln(r)\frac{\partial p}{\partial n}) d\theta \tag{7.11}$$

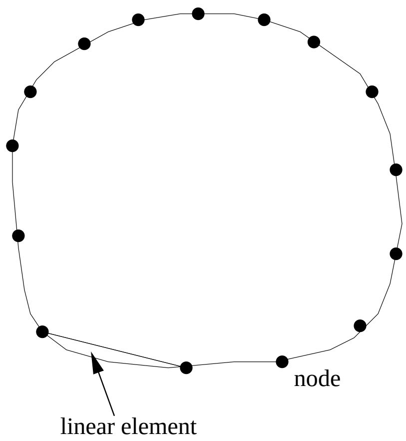

Figure 7.2: Boundary Discretisation

Consider the limit as $r \rightarrow 0$:

$$= \frac{1}{2\pi}(2\pi p) = p(x_i, y_i) \tag{7.12}$$

Now substitute this result back into equation 7.8:

$$p(x_i, y_i) + \int_\Gamma p\frac{\partial G}{\partial n}d\Gamma = \int_\Gamma G\frac{\partial p}{\partial n}d\Gamma \tag{7.13}$$

If the point (x_i, y_i) is on the boundary this equation must be modified to:

$$\frac{\theta}{2\pi}p(x_i, y_i) + \int_\Gamma p\frac{\partial G}{\partial n}d\Gamma = \int_\Gamma G\frac{\partial p}{\partial n}d\Gamma \tag{7.14}$$

where θ is the boundary angle (pi at smooth boundaries, $pi/2$ at right angle etc).

Note that at this stage no approximations have been introduced. The differential equation has been converted to an equivalent integral representation.

7.1.2 Boundary Discretisation

To obtain a numerical solution to the problem the boundary is divided into nodes and elements.

The geometry of the boundary and the variation of the unknown (pressure) is interpolated over the elements in terms of nodal values and shape

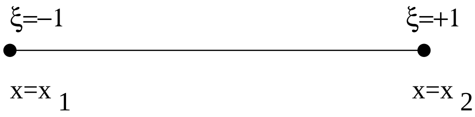

Figure 7.3: Local coordinate

functions (N_j). Figure 7.2 shows a linear element i.e. the boundary geometry is approximated by a straight line connecting the nodes. Higher order elements are also possible. To improve the accuracy of the solution the elements can either be refined into smaller elements, or higher order interpolation e.g. quadratic can be used.

The shape functions are defined in terms of a local coordinate ξ that runs from -1 to 1 along the element.

The linear shape functions are:

$$x(\xi) = N_1 x_1 + N_2 x_2 \tag{7.15}$$

$$y(\xi) = N_1 y_1 + N_2 y_2 \tag{7.16}$$

$$p(\xi) = N_1 p_1 + N_2 p_2 \tag{7.17}$$

where

$$N_1 = \frac{1}{2}(1-\xi) \tag{7.18}$$

$$N_2 = \frac{1}{2}(1+\xi) \tag{7.19}$$

The quadratic shape functions are:

$$x(\xi) = N_1 x_1 + N_2 x_2 + N_3 x_3 \tag{7.20}$$

$$N_1 = \frac{-\xi}{2}(1-\xi) \tag{7.21}$$

$$N_2 = (1+\xi)(1-\xi) \tag{7.22}$$

$$N_3 = \frac{\xi}{2}(1+\xi) \tag{7.23}$$

By breaking the boundary integral in equation 7.14 into the sum of integrals over elements, and writing the expression for the pressure and geometry of the elements in terms of shape functions we have:

$$\frac{\theta}{2\pi}p(x_i, y_i)+\sum_{m=1}^{M}\sum_{j=1}^{2}p_j\int_{-1}^{1}\frac{\partial G(r_i)}{\partial n}N_j(\xi)J(\xi)d\xi = \sum_{m=1}^{M}\sum_{j=1}^{2}\frac{\partial p_j}{\partial n}\int_{-1}^{1}G(r_i)N_j(\xi)J(\xi)d\xi \tag{7.24}$$

where M is the number of elements and $J(\xi)$ is a Jacobian which takes account of the integral being performed over ξ.

$$J(\xi) = \frac{d\Gamma}{d\xi} = \sqrt{\left(\frac{dx(\xi)}{d\xi}\right)^2 + \left(\frac{dy(\xi)}{d\xi}\right)^2} \tag{7.25}$$

The discretised integral equation can be written in matrix form as:

$$Ap = B\frac{\partial p}{\partial n} \tag{7.26}$$

Each row of this matrix equation arises from placing the source node of the Green's function at a given node i on the boundary. The terms in matrices A and B are usually too difficult to evaluate analytically.

$$A_{ij} = \int_{-1}^{1}\frac{\partial G(r_i)}{\partial n}N_j(\xi)J(\xi)d\xi + \frac{\theta}{2\pi}\delta_{ij} \tag{7.27}$$

$$B_{ij} = \int_{-1}^{1}G(r_i)N_j(\xi)J(\xi)d\xi \tag{7.28}$$

7.1.3 Gauss Quadrature

The following integral can be evaluated (approximately) using Gauss quadrature if $f(\xi)$ is not singular:

$$\int_{-1}^{1}f(\xi)d\xi = \sum_{n=1}^{N}f(\xi_n)W_n \tag{7.29}$$

The integral is evaluated as the weighted (weights, W_n) sum of function values at a set of special points known as Gauss points (ξ_n). The integral becomes more accurate as N increases.

$N = 2$

$$\pm\xi_n = 0.5773 \tag{7.30}$$

$$W_n = 1.0 \tag{7.31}$$

$N = 3$

$$\pm\xi_n = 0, 0.7745 \tag{7.32}$$

$$W_n = 0.8888, 0.5555 \tag{7.33}$$

$N = 4$

$$\pm\xi_n = 0.8611, 0.3399 \tag{7.34}$$

$$W_n = 0.3478, 0.6521 \tag{7.35}$$

If $i = j$ the integrals are singular special qaudratures can be used, however these may be computationally intensive or problem specific. Luckily a physical arguement can be used to evaluate these diagonal terms once the off-diagonals have been evaluated. For this problem if p is constant everywhere we expect $\frac{\partial p}{\partial n}$ at every node to be zero i.e.

$$Ap_{const} = B\frac{\partial p}{\partial n} = 0 \tag{7.36}$$

$$\Rightarrow A_{ii} = -\sum_{j=1, j\neq i}^{N} A_{ij} \tag{7.37}$$

7.1.4 Boundary Conditions

The solution is not uniquely determined until boundary conditions have been imposed. At every node pressure, pressure derivative or mixed boundary conditions must be imposed. Note that for the problem being considered at least one node must have a specified pressure. Once boundary conditions have been set half the unknown p and $\frac{\partial p}{\partial n}$ values can be eliminated from the matrix equation to give:

$$A'x = b \tag{7.38}$$

where x is a vector containing the unknown p and $\frac{\partial p}{\partial n}$ values.
At smooth boundaries either p or $q = \frac{\partial p}{\partial n}$ is set as a boundary condition:
At corners there are two values of q however. either
- specify both q values and solve for p
- specify one q value and p, then solve for the other q
- specify p and solve for both q values

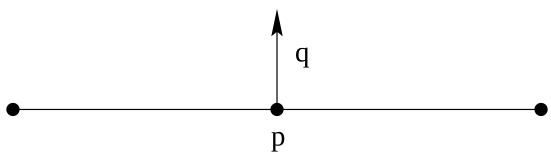

Figure 7.4: Boundary conditions at a smooth boundary

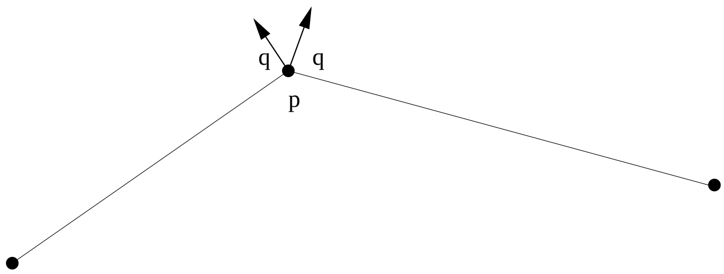

Figure 7.5: Boundary conditions at a corner

If the last option is being used and extra equation is required. This can be generated from the physics of the problem, or by breaking the boundary up.

7.1.5 Matrix Solution

The matrix A' is fully populated and has the same dimension as the number of boundary nodes. Direct solvers are usually used. The reduced size of the matrix involved in BEM is one of it's chief advantages. For instance, consider solving a problem one a 100 by 100 grid. If finite differences were being used the matrix would be 10,000 by 10,000. The matrix would be sparse. If BEM is used only the boundary nodes are required in the matrix problem so the matrix is 400 by 400, however this matrix would be dense.

7.1.6 Calculating Internal Solutions

The pressure can be calculated at any internal point (note there is no internal mesh) by placing the source point (x_i, y_i) of the Green's function at the point

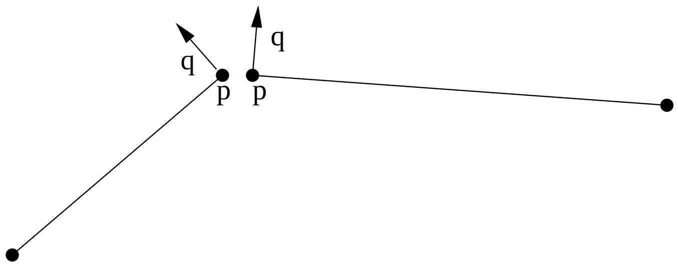

Figure 7.6: Broken boundary

of interest and revaluating the boundary integral equation.

$$\frac{\theta}{2\pi}p(x_i, y_i) = -\sum_{m=1}^{M}\sum_{j=1}^{2} p_j \int_{-1}^{1} \frac{\partial G(r_i)}{\partial n} N_j(\xi)J(\xi)d\xi + \sum_{m=1}^{M}\sum_{j=1}^{2} \frac{\partial p_j}{\partial n} \int_{-1}^{1} G(r_i)N_j(\xi)J(\xi)d\xi \tag{7.39}$$

Since we are considering internal points $\theta = 2\pi$. All the p_j and $\frac{\partial p_j}{\partial n}$ are known since they are located on the boundary and have already been determined. G is a function of (x_i, y_i) so the integrals must be revaluated.

Calculating the pressure at the internal points does not require a matrix solve, only revaluation of the element integrals. The internal solutions can be calculated sequentially, and only need to be calculated at the points of interest.

7.1.7 Transient Problems

Suppose we wanted to solve:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \eta \frac{\partial p}{\partial t} \tag{7.40}$$

If we begin from Green's second identity as before:

$$\iint_{\Omega} (f\nabla^2 g - g\nabla^2 f)d\Omega = \int_{\Gamma} f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}d\Gamma \tag{7.41}$$

We'll start by considering what happens when we divide Ω into a small circle surrounding the singularity and the remainder:

$$-\iint_{\Omega-\Omega_e} G\eta\frac{\partial p}{\partial t}d\Omega = \int_{\Gamma+\Gamma_e} p\frac{\partial G}{\partial n} - G\frac{\partial p}{\partial n}d\Gamma \tag{7.42}$$

since $\nabla^p = \eta\frac{\partial p}{\partial t}$. We're using the steady state Green's function used in the previous section.

We could approximate $\frac{\partial p}{\partial t}$ with a finite difference i.e.:

$$\frac{\partial p}{\partial t} = \frac{p^{t2} - p^{t1}}{\Delta t} \tag{7.43}$$

however a domain integral is still required so the problem has lost it's boundary only character.

To handle transient problems in a boundary only manner alternative approaches are require. We'll consider two - transient Green's functions and solving the problem in Laplace space.

7.1.8 Transient Problems - Transient Green's Functions

We've already discussed how to find Green's functions for equations of the form:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \eta\frac{\partial p}{\partial t} \tag{7.44}$$

.

If we have the correct Green's functions we can build a boundary only solution method of the form:

$$Ap = Bq \tag{7.45}$$

The procedure to do so involves multiplying the governing differential by G and integrating by parts. The A_{ij} and B_{ij} terms are now time dependent so the element integrals must be reevaluated each time step. Computationally efficient ways to do so are an ongoing research topic.

7.1.9 Transient Problems - Laplace Transform

If the initial pressure is zero then the Laplace transform of the differential equation is:

$$\frac{\partial^2 \bar{p}}{\partial x^2} + \frac{\partial^2 \bar{p}}{\partial y^2} = s\bar{p} \tag{7.46}$$

The Green's function is available (in Laplace space) for this problem:

$$G = \frac{1}{2\pi}K_o(\sqrt{s}r) \tag{7.47}$$

Using this Green's function a BEM scheme can be built which will solve for $\bar{p}$. To get p at any time of interest we can numerically invert this solution. This removes the need to begin the solution procedure from $t = 0$ and march forward in time.

7.1.10 Advantages of BEM

- No internal grid. The only discretisation errors come from the boundary discretisation. Internal solutions are actually more accurate than the boundary solutions. No grid orientation effect.
- Flexible geometry. Good boundary conformance.
- Reduction of dimensionality i.e. one dimensional grid required for a two dimensional problem, two dimensional grid required for a three dimensional problem. Smaller matrix to solve.

7.1.11 Disadvantages of BEM

- Only applicable to problems with a ∇^2 operator e.g. diffusion equation, wave equation. Problems with convective terms can not be handled. For reservoir engineering problems only applicable to single phase flow in homogeneous media.
- Transient problems are more complicated than steady-state.
- Dense matrix problem.

7.2 Alternatives to the Boundary Element Method

BEM is only applicable to problems we can find the Green's function for. This limits it's applicability. The theory can be extended to a wider class of problems by considering alternative boundary element based methods including the Dual Reciprocity Boundary Element Method (DRBEM) and the Green Element Method (GEM). DRBEM is the more commonly used approach however my Ph.D. research suggests GEM has some attractive properties for reservoir engineering problems. The two approaches will be introduced in the following sections.

7.2.1 Dual Reciprocity Boundary Element Method

Suppose we want to solve a problem of the form:

$$\nabla^2 p = b(x, y) \tag{7.48}$$

The function b is arbitrary so the Green's function for this problem is not neccessarily available. The essence of the DRBEM approach is the following expansion:

$$b \approx \sum_{j=1}^{N} \alpha_j f_j \tag{7.49}$$

where the α_j are weights and the f_j are a set of approximating functions. The only restriction on the approximating functions is that they are the Laplacian of some other function:

$$\nabla^2 \hat{p_j} = f_j \tag{7.50}$$

One of the simplest functions satisfying this requirement is $f = 1 + r$. The equation we are trying to solve can now be expressed as:

$$\nabla^2 p = \sum_{j=1}^{N} \alpha_j f_j = \sum_{j=1}^{N} \alpha_j \nabla^2 \hat{p_j} \tag{7.51}$$

This introduces a ∇^2 operator on each side of equation so well can apply the same procedure we used for the $\nabla^2 p = 0$ problem to derive a boundary integral equation:

$$\frac{\theta}{2\pi} p(x_i, y_i) + \int_\Gamma \frac{\partial G}{\partial n} p d\Gamma - \int_\Gamma G \frac{\partial p}{\partial n} = \sum_{j=1}^{N} \alpha_j \left(\frac{\theta}{2\pi} \hat{p_j}(x_i, y_i) + \int_\Gamma \frac{\partial G}{\partial n} \hat{p_j} - \int_\Gamma G \frac{\partial \hat{p_j}}{\partial n}\right) \tag{7.52}$$

This can be written in matrix form as:

$$Ap - Bq = (A\hat{P} - B\hat{Q})\alpha \tag{7.53}$$

where $\hat{P}$ and $\hat{Q}$ are matrices whose columns contain the functions p_j and q_j evaluated at each node i.

Computational issues:

- internal nodes must be included in the matrix problem if the solution is required at internal points. So the matrix problem is larger than it would be for BEM and it remains dense.

- the weighting vector α may require a significant amount of calculation. In transient problems this vector must be updated at each timstep.
- DRBEM does extend the theory of BEM to arbitrary problems (as long as they include a ∇^2 operator somewhere. Convective problems, such as the convection-diffusion equation can be considered using DRBEM.

7.2.2 The Green Element Method

The Green Element was derived by Professor Akpofure Taigbenu and is fully explained is his book "The Green Element Method". It's chief attraction is it's flexibility and the fact that it generates a sparse matrix problem. The matrix problem is however large.
Suppose we want to solve:

$$\nabla^2 p = b(x, y) \tag{7.54}$$

Again we will start from Green's second identity:

$$\iint_\Omega p\nabla^2 G - G\nabla^2 p d\Omega = \int_\Gamma p\frac{\partial G}{\partial n} - G\frac{\partial p}{\partial n} d\Gamma \tag{7.55}$$

Substituing $\nabla^2 p = b$ and splitting the domain into a circle surrounding the singularity and the remainder (as before) gives:

$$-\frac{\theta}{2\pi} p(x_i, y_i) + \int_\Gamma p\frac{\partial G}{\partial n} - G\frac{\partial p}{\partial n} d\Gamma = \iint_\Omega G b d\Omega \tag{7.56}$$

Note that no approximations have been made at this stage. Now GEM departs from BEM:
- both the boundary and the domain are discretised.
- the boundary integral can be seen as the sum of integrals over the element boundaries.

If both the boundary and the domain integrals are broken into the sum of element integrals we have:

$$-\frac{\theta}{2\pi} p(x_i, y_i) + \sum_{e=1}^{M} \int_{\Gamma_e} p\frac{\partial G}{\partial n} - G\frac{\partial p}{\partial n} d\Gamma = \sum_{e=1}^{M} \iint_{\Omega_e} G b d\Omega \tag{7.57}$$

Now shape functions are introduced to interpolate the pressure and $q = \frac{\partial p}{\partial n}$ over the elements in terms of nodal values. In the case of linear shape

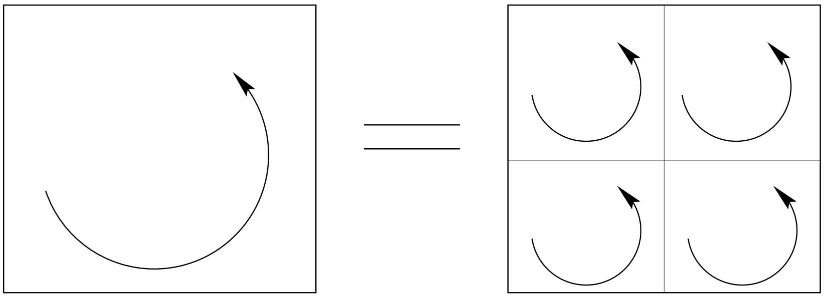

Figure 7.7: Overall boundary is equivalent to the sum of the element boundaries

functions:

$$p = \sum_{j=1}^{4} N_j p_j \tag{7.58}$$

$$q = \sum_{j=1}^{4} N_j q_j \tag{7.59}$$

$$b = \sum_{j=1}^{4} N_j b_j \tag{7.60}$$

When these shape functions are substituted into (7.57) we have:

$$-\frac{\theta}{2\pi} p(x_i, y_i) + \sum_{e=1}^{M} \sum_{j=1}^{4} \int_{\Gamma_e} p_j N_j \frac{\partial G}{\partial n} - G q_j N_j d\Gamma = \sum_{e=1}^{M} \sum_{j=1}^{4} \iint_{\Omega_e} G b_j N_j d\Omega \tag{7.61}$$

A matrix equation can be formed from this:

$$\sum_{e=1}^{M} R_{ij}^e p_j + L_{ij}^e q_j + V_{ij} b_j = 0 \tag{7.62}$$

where

$$R_{ij}^e = \int_{\Gamma_e} \frac{\partial G}{\partial n} N_j d\Gamma \tag{7.63}$$

$$L_{ij}^e = \int_{\Gamma_e} G_i N_j d\Gamma \tag{7.64}$$

$$V_{ij}^e = \iint_{\Omega_e} G_i N_j d\Omega \tag{7.65}$$

7.2.3 Dealing with Heterogeneity

Single phase flow in heterogeneous reservoirs is governed by:

$$\nabla \cdot (\frac{k}{\mu}\nabla p) = \phi c_t \frac{\partial p}{\partial t} \tag{7.66}$$

This is not in a form suitable for solution with any BEM based method. However a ∇^2 operator can be extract by rearranging the equation in the following manner:

$$\nabla^2 p = -\nabla lnk \cdot \nabla p + \frac{\phi \mu c_t}{k}\frac{\partial p}{\partial t} \tag{7.67}$$

Chapter 8

Errata

The following is a list of typos that were found after the original handout was produced. There are inevitably even more. If you find them please let me know.

Eqn 2.20 should read $\lim s \to \infty \;\; sLf(t)$, the first = sign is incorrect
Eqn 2.22 should have $\lim s \to \infty$ as the second limit
Eqn 2.63 should not have the first negative sign
Eqn 2.66 should have limits of integration from s to infinity
Eqn 2.81 should have $\lim t \to \infty$ on the right hand side
Eqn 3.48 should be $\lim t \to \infty$
Eqn 3.73 should be pd = (pi-p)/(pi-pw)
Eqn 3.74 should be rd=1
Eqn 6.82 should finish with $\delta(x_i - x)dx_i$
Eqn 6.84 should have 2B exp(2x) not B exp(2x)
Eqn 6.86 should finish with $x \leq xi \leq 1$
Eqn 6.171 should have $-z^2/A^2$ as the exponent