# PE281 Boundary Element Method Course Notes

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## 1 Background Theory

The idea of boundary element methods is that we can approximate the solution to a PDE by looking at the solution to the PDE on the boundary and then use that information to find the solution inside the domain. This sounds like a strange idea, but it is a very powerful tool for finding solutions. In particular BEM is useful on very large domains where a FEM approximation would have too many elements to be practical.

A PE example of where BEM would be superior to FEM is a series of wells in an infinite reservoir. In order to use FEM the reservoir would have to be gridded to some large distance from all of the wells so that boundary has little impact on the pressure solution at the wells. There are two disadvantages here: First of all how far from the wells is "far enough?" Do we need 10 gridblocks beyond the wells near the edge to get an accurate solution? 100? 1000? Second of all, the FEM approximation would probably involve an extremely large mesh in which most of the elements are far from the wells. So this method would spend a lot of time computing approximations in the gridblocks far from the wells, which is not generally the part of the domain we are interested in. In order to apply BEM to this problem we would only have to create a mesh on the boundary of the domain. For an infinite reservoir the only boundaries are the wells. This means that we would spend all of our time finding the solution at the wells, which IS usually the part of the domain that we are interested in. Moreover, since the BEM mesh only

approximates the pressure solution at the wells the domain is small and this is a computationally inexpensive problem.

| FEM | BEM |
|---|---|
| discretization of whole domain | discretization of boundary |
| good on finite domains | good on infinite or semi-infinite domains |
| approximates $u$, $q$ must be found from $u$ and may not be as accurate | approximates $q$ and $u$ approximation of $q$ is accurate |
| large, sparse matrix $Ku = F$ | small, filled-in matrix $Hu = Gq$ |
| requires no prior knowledge of solution | requires a fundamental solution to the PDE |
| solves most linear second-order PDEs | can be difficult to solve inhomogenerous or nonlinear problems |

## 1.1 Linear Differential Operators

The idea of linear operators is one that we have been using informally for most of the class. A linear differential operator, denoted $L$ is a function such that

$$L = \frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots + p_{n-1}(x)\frac{d}{dx} + p_n(x) \tag{1}$$

in one dimension or

$$L = \frac{\partial^n}{\partial x^n} + p_1(x,y)\frac{\partial^{n-1}}{\partial x^{n-1}} + q_0(x,y)\frac{\partial^n}{\partial y^n} + q_1(x,y)\frac{\partial^{n-1}}{\partial y^{n-1}} + r_1(x,y)\frac{\partial^{n-1}}{\partial x^m \partial y^{n-1-m}} + \ldots + r_{n-2}(x,y)\frac{\partial^2}{\partial x \partial y} + p_{n-1}(x,y)\frac{\partial}{\partial x} + q_{n-1}(x,y)\frac{\partial}{\partial y} + p_n(x,y) \tag{2}$$

in two-dimensions.

All linear operators have the property that $L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$ where $\alpha$ and $\beta$ are constants and $u$ and $v$ are functions.

The *adjoint operator* $L^*$ is the function that makes

$$\int_\Omega L(u)\, v d\Omega = \int_\Omega u L^*(v)\, d\Omega. \tag{3}$$

Fortunately many operators are *self-adjoint* so that $L = L^*$. Even if $L$ is not self-adjoint, usually $L$ and $L^*$ are similar functions that differ by a $\pm$ sign in one or more terms.

### 1.1.1 Example of a Self-Adjoint Operator

Let $L = \frac{d^2}{dx^2} + 1$ on the interval $[0, 1]$. $L(u) = \frac{d^2u}{dx^2} + u$. Now

$$\begin{array}{l} \int_0^1 \frac{d^2u}{dx^2} w + uw dx = \frac{du}{dx} w\Big|_0^1 - \int_0^1 \frac{du}{dx}\frac{dw}{dx} dx + \int_0^1 uw dx \\ = \frac{du}{dx} w\Big|_0^1 - \frac{dw}{dx} u\Big|_0^1 + \int_0^1 \frac{d^2w}{dx^2} u dx + \int_0^1 uw dx = \\ boundary\ terms + \int_\Omega uL(v)\, d\Omega \end{array} \tag{4}$$

Since the function $w$ was arbitrary we can assume that $w$ and $\frac{dw}{dx}$ are both zero at $x = 0$ and $x = 1$ so that the boundary terms drop out. As a consequence $\int_0^1 L(u)\, v d\Omega = \int_0^1 uL(v)\, d\Omega$, so this operator is self-adjoint.

## 1.2 The Fundamental Solution

Consider the Laplace euqation in two-dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{5}$$

along with some boundary conditions. The *fundamental solution* or *Freespace Green's Function* satisfies the equation

$$\frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} + \delta(\xi - x, \eta - y) = 0 \tag{6}$$

on the domain $-\infty < x < \infty$, $-\infty < y < \infty$. $\delta$ is the dirac-delta function in two-dimensions. ie a point source of infinite strength at $(\xi, \eta)$.

The fundamental solution of any PDE is the analytical solution of the governing PDE under action of a point source and on an infinite domain. The fundamental solutions of most common linear operators are well known, but it can be very challenging to find the fundamental solution for inhomogeneous, non-linear or anisotropic operators.

In BEM the fundamental solution will be used as the weighting function in the integral equation and is analgous to the test functions $v$ in finite elements.

Transforming the Laplace equation to radial coordinates where the radius $r$ is measured from the point $(\xi, \eta)$.

$$\nabla^2 w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = 0 \tag{7}$$

and

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2}. \tag{8}$$

The $\delta$ term has disappeared from the equation because it is only nonzero when $r = 0$, so now the delta function is acting as a boundary condition that says that $w \to \infty$ as $r \to 0$. The $\frac{\partial^2 w}{\partial \theta^2}$ term is also zero because we have an infinite domain so the final equation is $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = 0$ which has the solution

$$w = A \ln r + B. \tag{9}$$

This solution satisfies the boundary condition at $r = 0$ since $\ln r \to \infty$ as $r \to 0$, but so far we have no information on how to find $A$ and $B$.

Looking back at the original equation in cartesian coordinates we can integrate so that

$$\int_\Omega \nabla^2 w d\Omega + \int_\Omega \delta\left(\xi - x, \eta - y\right) d\Omega = 0 \tag{10}$$

if $(\xi, \eta) \in \Omega$ then $\int_\Omega \delta\left(\xi - x, \eta - y\right) d\Omega = 1$. Since $\Omega$ is arbitrary, we will assume that $(\xi, \eta) \in \Omega$. This means that $\int_\Omega \nabla^2 w d\Omega = -1$. Since $\Omega$ is arbitrary we can assume that it is a circle of radius $\varepsilon > 0$ centered at $r = 0$. Now by the Green-Gauss theorem (see Kreyszig p 490)

$$\int_\Omega \nabla^2 w d\Omega = \int_{\partial\Omega} \frac{\partial w}{\partial n} ds \tag{11}$$

where $n$ is the normal vector and $s$ is the distance along the path counter-clockwise. Since $r$ and the normal vector $n$ will always point in the same direction when $\Omega$ is a circle this is equivalent to

$$\int_{\partial\Omega} \frac{\partial w}{\partial r} ds \tag{12}$$

but since we know that $w = A \ln r + B$ we can find $\frac{\partial w}{\partial r} = \frac{A}{r} = \frac{A}{\varepsilon}$ and we have that

$$\int_0^{2\pi\varepsilon} \frac{A}{\varepsilon} ds = \frac{A\,(2\pi\varepsilon)}{\varepsilon} = -1 \tag{13}$$

so $A = \frac{-1}{2\pi}$. Since we have no good way to find $B$, it is usually set to zero for convenience. The fundamental solution for the Laplace equation is

$$w = \frac{-1}{2\pi} \ln r. \tag{14}$$

# 2 Derivation of the Boundary Element Method in 2D

Exactly like in the finite element method we are trying to solve a PDE by using a weighted integral equation. In this example we will look at the Laplace equation, but BEM can be derived for any PDE for which we can find a fundamental solution.

Once again we begin by multiplying the PDE by a weighting function $w$ and integrate over the domatin to find a weak solution:

$$\nabla^2 u = 0 \Rightarrow w\left[\nabla^2 u = 0\right] \Rightarrow \int_\Omega \left[\nabla^2 u w\right] d\Omega = 0 \tag{15}$$

Using the Green-Gauss theorem gives

$$\int_\Omega \left[\nabla^2 uw\right] d\Omega = \int_{\partial\Omega} \frac{\partial u}{\partial n} w ds - \int_\Omega \nabla u \nabla w d\Omega. \tag{16}$$

Integrating by parts again in order to get a second derivative on the weighting function $w$:

$$\int_\Omega \left[\nabla^2 uw\right] d\Omega = \int_{\partial\Omega} \frac{\partial u}{\partial n} w ds - \int_{\partial\Omega} \frac{\partial w}{\partial n} n ds + \int_\Omega u \nabla^2 w d\Omega \tag{17}$$

This shows that the Laplace operator is self adjoint.

In FEM we chose simple piecewise polynomials as our weighting (test) functions. In BEM we will choose the fundamental solution so that the last term becomes

$$\int_\Omega u \nabla^2 w d\Omega = -\int_\Omega u\delta\left(\xi - x, \eta - y\right) d\Omega = -u\left(\xi, \eta\right) \tag{18}$$

assuming that $(\xi, \eta) \in \Omega$ and not on the boundary. This gives us the *boundary integral equation*

$$-\int_{\partial\Omega} \frac{\partial u}{\partial n} w ds + \int_{\partial\Omega} \frac{\partial w}{\partial n} n ds + u\left(\xi, \eta\right) = 0 \tag{19}$$

for $(\xi, \eta) \in \Omega$. This means that we can (in theory) find $u$ at an arbitrary point $(\xi, \eta) \in \Omega$ by looking at $u$ and $w$ only on the boundary. This still doesn't help us unless we know $u$ and/or $\frac{\partial u}{\partial n}$ on the boundary. In order to solve this problem we will look at what happens if $(\xi, \eta)$ is on $\partial\Omega$.

First, what happens if $(\xi, \eta) \notin \Omega$:

$$\int_\Omega u \nabla^2 w d\Omega = -\int_\Omega \delta\left(\xi - x, \eta - y\right) d\Omega = 0 \tag{20}$$

because $\delta$ is zero everywhere in $\Omega$.

Assume that some point $P = (\xi, \eta)$ is on $\partial\Omega$. Define a disk with radius $\varepsilon$ around $P$. Now we have two subdomains on the boundary $\partial\Omega = \partial\Omega_{-\varepsilon} + \partial\Omega_\varepsilon$. We want to take the limit of each domain as $\varepsilon \to 0$. In order to do this we will have to look at four integrals:

$$\int_{\partial\Omega_\varepsilon} u\frac{\partial w}{\partial n}ds = \int_{\partial\Omega_\varepsilon} u\frac{\partial}{\partial n}\left(\frac{-1}{2\pi}\ln r\right)ds = \int_{\partial\Omega_\varepsilon}\frac{-u}{2\pi r}ds \tag{21}$$

$$= \frac{-1}{2\pi r}\int_{\partial\Omega_\varepsilon} uds = -u\left(P\right)\frac{\pi\varepsilon}{2\pi} = \frac{-u\left(P\right)}{2} \tag{22}$$

where $r = \varepsilon$ is fixed since the domain is a circle and in the limit of the integral is the value at $P$ by the mean value theorem. Similarly

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} w\frac{\partial u}{\partial n}ds = \frac{-\ln\varepsilon}{2\pi}\int_{\partial\Omega_\varepsilon}\frac{\partial u}{\partial n}ds = \frac{-\ln\varepsilon}{2\pi}\frac{\partial u\left(P\right)}{\partial n}\pi\varepsilon \to 0 \tag{23}$$

For the two integrals over $\partial\Omega_{-\varepsilon}$ we have that

$$\begin{array}{l}\lim_{\varepsilon\to 0}\int_{\partial\Omega_{-\varepsilon}} w\frac{\partial u}{\partial n}ds = \int_{\partial\Omega} w\frac{\partial u}{\partial n}ds \\ \lim_{\varepsilon\to 0}\int_{\partial\Omega_{-\varepsilon}} u\frac{\partial w}{\partial n}ds = \int_{\partial\Omega} u\frac{\partial w}{\partial n}ds\end{array} \tag{24}$$

Putting this all together gives

$$u\left(P\right) + \int_{\partial\Omega} u\frac{\partial w}{\partial n}ds = \frac{1}{2}u\left(P\right) + \int_{\partial\Omega} w\frac{\partial u}{\partial n}ds \tag{25}$$

$$\frac{1}{2}u\left(P\right) + \int_{\partial\Omega} u\frac{\partial w}{\partial n}ds = \int_{\partial\Omega} w\frac{\partial u}{\partial n}ds \tag{26}$$

for $u(P)$ on $\partial\Omega$ and $\partial\Omega$ smooth at $P$. If $\partial\Omega$ is not smooth at $P$ then

$$\left(1 - \frac{\alpha}{2\pi}\right)u\left(P\right) + \int_{\partial\Omega} u\frac{\partial w}{\partial n}ds = \int_{\partial\Omega} w\frac{\partial u}{\partial n}ds \tag{27}$$

where $\alpha$ is the interior angle of the corner at $P$. In general

$$c\left(P\right)u\left(P\right) + \int_{\partial\Omega} u\frac{\partial w}{\partial n}ds = \int_{\partial\Omega} w\frac{\partial u}{\partial n}ds \tag{28}$$

$$c\left(P\right) = \begin{cases} 1 & P \in \Omega \\ 1/2 & P \in \partial\Omega, \ \ \partial\Omega \ smooth \\ \left(1 - \frac{\alpha}{2\pi}\right) & P \in \partial\Omega, \ \ \partial\Omega \ not \ smooth \\ 0 & P \notin \Omega \end{cases} \tag{29}$$

$$\tag{30}$$

# 3 Putting it All Together

In order to formulate BEM we need to divide the 2D domain up into $N$ boundary elements or subdomains $\Gamma_i$. We will make a number of assumptions about the elements and the functions on the elements:

- We will be given either $u$ or $q = \frac{\partial u}{\partial n}$ on a given element $\Gamma_i$. BEM will be used to find the other.
- Each element has one node at its center.
- We will assume that both $u$ and $q$ are constant on each element, even though the exact function $q = \frac{\partial u}{\partial n}$ would have a lower order than $u$. This is called an *isoparametric* formulation.
- Like in FEM it is possible to define nodes at the end of the elements and use linear functions, or to define three nodes per element and use quadratics. We will constrict our focus to constant elements because they are simplest.
- Because each element has only one node for constant elements $u$ and $q$ are not continuous on the boundary.
- The boundary is always smooth for constant elements since we are always evaluating the function at the midpoint of the line connecting the two edges of the element. As a consequence $c(P) = 0.5$.

Consider the point $i$ on the boundary:

$$\frac{1}{2}u_i + \int_{\partial\Omega} u\frac{\partial w}{\partial n}ds = \int_{\partial\Omega} w\frac{\partial u}{\partial n}ds \tag{31}$$

$$\frac{1}{2}u_i + \sum_{j=1}^{N}\left[\int_{\Gamma_j} uq^* d\Gamma\right] = \sum_{j=1}^{N}\left[\int_{\Gamma_j} wq d\Gamma\right] \tag{32}$$

$$\tag{33}$$

where $q^* = \frac{\partial w}{\partial n}$. Since $u_j$ and $q_j$ are constant on each element

$$\frac{1}{2}u_i + \sum_{j=1}^{N} u_j \left[\int_{\Gamma_j} q^* d\Gamma\right] = \sum_{j=1}^{N} q_j \left[\int_{\Gamma_j} w d\Gamma\right]. \tag{34}$$

This gives two integrals that we need to approximate:

$$\hat{H}_{ij} = \int_{\Gamma_j} q^* d\Gamma \;\text{ and }\; G_{ij} = \int_{\Gamma_j} w d\Gamma. \tag{35}$$

these integrals give the influence of some node $j$ on the fundamental solution that is centered at $i$ on the boundary and are called *influence coefficients*. For a single point $i$ on the boundary

$$\frac{1}{2}u_i + \sum_{j=1}^{N} u_j \hat{H}_{ij} = \sum_{j=1}^{N} q_j G_{ij} \tag{36}$$

so $u_i = u_i(u_1, u_2, \ldots, u_N, q_1, q_2, \ldots, q_N)$. Since half of all of the $u_i$ and $q_i$ are specified by doundary conditions we need $N$ equations like Eq. 36. This is accomplished by moving the singular point $i$ around the boundary and evaluating at every node $i = 1, \ldots, N$. For simplicity define

$$H_{ij} = \begin{cases} \hat{H}_{ij} & i \neq j \\ \hat{H}_{ij} + 0.5 & i = j \end{cases} \tag{37}$$

Now we have $N$ equations

$$\sum_{j=1}^{N} u_j H_{ij} = \sum_{j=1}^{N} q_j G_{ij} \tag{38}$$

Which becomes the matrix equation $HU = GQ$ where $H$ and $G$ are $N \times N$ matricies and $U$ and $Q$ are vectors of length $N$:

$$\begin{bmatrix} H_{11} & H_{12} & \ldots & H_{1N} \\ H_{21} & H_{22} & \ddots & H_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ H_{N1} & H_{N2} & \ldots & H_{NN} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & \ldots & G_{1N} \\ G_{21} & G_{22} & \ddots & G_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ G_{N1} & G_{N2} & \ldots & G_{NN} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{bmatrix} \tag{39}$$

This formulation of the problem presents a problem since $u$ is given in some places and $q$ is given in others. In order to solve the linear equation we will have to get all of the unknowns on the left hand side of the equation and the knowns on the right hand side. This is accomplished by shuffling the columns of the matrix.

The easiest way to explain this is by using an example. Suppose that the boundary is divided up into 12 elements. The value of $u$ is given on each of the first six boundary elements so that $u_i = \hat{u}_i$ for $i = 1, \ldots, 6$. The value of 1 is given on elements seven through 12 so that $q_i = \hat{q}_i$ for $i = 7, \ldots, 12$.

$$\begin{aligned} &\begin{bmatrix} H_{1,1} & \ldots & H_{1,6} & H_{1,7} & \ldots & H_{1,12} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ H_{12,1} & \ldots & H_{12,6} & H_{12,7} & \ldots & H_{12,12} \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \vdots \\ \hat{u}_6 \\ u_7 \\ \vdots \\ u_{12} \end{bmatrix} \\ &= \begin{bmatrix} G_{1,1} & \ldots & G_{1,6} & G_{1,7} & \ldots & G_{1,12} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ G_{12,1} & \ldots & G_{12,6} & G_{12,7} & \ldots & G_{12,12} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_6 \\ \hat{q}_7 \\ \vdots \\ \hat{q}_{12} \end{bmatrix} \end{aligned} \tag{40}$$

Now $u_7$ to $u_{12}$ and $q_1$ to $q_6$ are the unknowns. When they are shuffled to the left hand side of the equation we get

$$\begin{bmatrix} G_{1,1} & \dots & G_{1,6} & -H_{1,7} & \dots & -H_{1,12} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ G_{12,1} & \dots & G_{12,6} & -H_{12,7} & \dots & -H_{12,12} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_6 \\ u_7 \\ \vdots \\ u_{12} \end{bmatrix}$$
$$= \begin{bmatrix} H_{1,1} & \dots & H_{1,6} & -G_{1,7} & \dots & -G_{1,12} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ H_{12,1} & \dots & H_{12,6} & -G_{12,7} & \dots & -G_{12,12} \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \vdots \\ \hat{u}_6 \\ \hat{q}_7 \\ \vdots \\ \hat{q}_{12} \end{bmatrix} \tag{41}$$

which is a matrix equation of the form $Ax = f$. Now we can find $q$ and $u$ at every point on the boundary by multiplying $x = A^{-1}f$. Notice that $A$ is a dense matrix, which makes this a more computationally expensive matrix to solve than in FEM. The advantage is that we are solving a 1D problem on the boundary rather than a 2D problem, so this is a smaller, denser matrix than we would get from FEM.

In order to find the value of $u$ at some point in the interior of the domain we go back to Eqs. 28-30 to see that

$$u_i = \int_\Gamma wqd\Gamma - \int_\Gamma uq^* d\Gamma \tag{42}$$

which simplifies to

$$u_i = \sum_{j=1}^{N} q_j G_{ij} - \sum_{j=1}^{N} u_j \hat{H}_{ij} \tag{43}$$

Frequently we are not only interested in the value of $u$ at point $i$, but also the flux in the $x$ or $y$ direction. We can find the flux in the $x$ direction from

$$q_{ix} = \frac{\partial u}{\partial x} = \int_\Gamma \left(\frac{\partial w}{\partial x}\right)_i qd\Gamma - \int_\Gamma u \left(\frac{\partial q^*}{\partial x}\right)_i d\Gamma \tag{44}$$

where there are no partial derivatives of $u$ and $q$ because they are constants along each element. Putting in the element mesh gives

$$q_i = \sum_{j=1}^{N} q_j \int_{\Gamma_j} \left( \frac{\partial w}{\partial x} \right) d\Gamma - \sum_{j=1}^{N} u_j \int_{\Gamma} \left( \frac{\partial q^*}{\partial x} \right) d\Gamma. \tag{45}$$

Notice that because we are looking at the x-direction flux, so the term $\frac{\partial q^*}{\partial x}$ is a mixed partial derivative.

Typically all of the integrals in BEM integrated numerically using quadrature rules, just like in FEM. Since in BEM the integrals have complex functions inside, it is typically necessary to use high-order approximations.

# References


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