PE281 Green's Functions Course Notes

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Stanford, CA 7th June 2006

1 What are Green's Functions?

Recall that in the BEM notes we found the fundamental solution to the Laplace equation, which is the solution to the equation

$$\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \delta\left(\xi - x, \eta - y\right) = 0 \tag{1}$$

on the domain $-\infty < x < \infty$, $-\infty < y < \infty$. δ is the dirac-delta function in two-dimensions. This was an example of a Green's Fuction for the two-dimensional Laplace equation on an infinite domain with some prescribed initial or boundary conditions. The difference between BEM and the method of Green's functions is that we will be looking at PDEs that are sufficiently simple to evaluate the boundary integral equation analytically.

The PDE we are going to solve initially is

$$\nabla^2 u = 0 \tag{2}$$

$$u|_{\partial\Omega} = f\left(x, y\right) \tag{3}$$

As in BEM we will start by applying the Green-Gauss Theorem two times to get

$$\left(L\left(u\right), G\right) = \int_{\Omega} G\nabla^2 u d\Omega = \int_{\partial\Omega}\left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\right) ds + \int_{\Omega} u\nabla^2 G d\Omega = 0 \tag{4}$$

where n is the outward pointing normal and ds is the distance along the boundary, counterclockwise. Since u is given on the boundary and $\frac{\partial u}{\partial n}$ is not, we will set $G = 0$ on $\partial\Omega$.

$$\int_{\partial\Omega} \left(-f\left(x, y\right) \frac{\partial G}{\partial n} \right) ds + \int_{\Omega} u \nabla^2 G d\Omega = 0 \tag{5}$$

Using the definition of the Green's function $\nabla^2 G d\Omega = -\delta(\xi - x, \eta - y)$ gives

$$u\left(\xi, \eta\right) = -\int_{\partial\Omega} \left(f\left(x, y\right) \frac{\partial G}{\partial n} \right) ds \tag{6}$$

Some interesting things about this formulation

- We have a negative sign in front of the boundary integral, this is because we defined the point source so that $-u(\xi, \eta) = \delta(\xi - x, \eta - y)$. This sign convention is opposite what is used in Ruben's notes, but it has no impact on the final answer.
- We can now find u at some arbitrary point (ξ, η) just by integrating $f \frac{\partial G}{\partial n}$ around the boundary of the domain.
- Because we are using the Green's function for this specific domain with Dirichlet boundary conditions, we have set $G = 0$ on the boundary in order to drop one of the boundary integral terms.
- The fundamental solution is not the Green's function because this domain is bounded, but it will appear in the Green's function.
- We want to seek $G(\xi, \eta; x, y) = w + g$ where w is the fundamental solution and does not satisfy the boundary constraints and g is some function that is zero in the domain and will allow us to satisfy the boundary conditions.
- We know from BEM notes that $w = \frac{-1}{2\pi} \ln r$ where $r = \sqrt{\left(\xi - x\right)^2 + \left(\eta - y\right)^2}$

2 Example of Laplace's Equation

Suppose the domain is the upper half-plan, $y > 0$. We know that $G = \frac{-1}{2\pi} \ln r + g$ and that g must satisfy the constraint that $\nabla^2 g = 0$ in the domain $y > 0$ so that the Green's function supplies a single point source in the real domain. We also know that along the line $y = 0$ $g = -w$ so that $G(x, 0) = 0$. This equation can be solved by the method of images. This means that we will introduce point sources outside of the domain to satisfy the boundary conditions.

In this example a negative point source at $(\xi, -\eta)$ will give $g = -w$ on $G(x, 0) = 0$. The solution to $\nabla^2 g = \delta(\xi - x, \eta + y)$ will also satisfy the constraint that $\nabla^2 g = 0$ since $\nabla^2 g$ is nonzero only at the point source $(\xi, -\eta)$, which is not in the domain.

$$g = \frac{1}{2\pi} \ln \sqrt{(\xi - x)^2 + (\eta + y)^2} \tag{7}$$

Now

$$\begin{aligned} G &= g + w \quad (8) \\ &= \frac{1}{2\pi} \ln \sqrt{(\xi - x)^2 + (\eta + y)^2} - \frac{1}{2\pi} \ln \sqrt{(\xi - x)^2 + (\eta - y)^2} \quad (9) \\ &= \frac{1}{4\pi} \ln \left(\frac{(\xi - x)^2 + (\eta + y)^2}{(\xi - x)^2 + (\eta - y)^2} \right) \quad (10) \end{aligned}$$

so

$$u(\xi, \eta) = -\int_{-\infty}^{\infty} \left(f(x) \frac{\partial G}{\partial n}(x, 0) \right) dx \tag{11}$$

this can be multiplied out to show that

$$\begin{aligned} &u(\xi, \eta) = \quad (12) \\ &\int_{-\infty}^{\infty} \left(\frac{1}{2\pi} f(x) \frac{\eta \left[(\xi - x)^2 + \eta^2 \right]^{-1/2} + \eta \left[(\xi - x)^2 + \eta^2 \right]^{-1/2}}{\left[(\xi - x)^2 + \eta^2 \right]^{-1/2}} \right) dx \quad (13) \end{aligned}$$

$$= \frac{\eta}{\pi} \int_{-\infty}^{\infty} \left(\frac{f(x)}{\left[(\xi - x)^2 + \eta^2 \right]} \right) dx. \tag{14}$$

We could have gotten this solution by Fourier transforms, but Green's functions can also be used to find solutions for many problems that can't be solved by transform methods.

3 Example of Poisson's Equation

Now we will look at Poisson's equation on the same domain, which is given by

$$\nabla^2 u = \phi(x, y) \tag{15}$$

$$u|_{\partial\Omega} = f(x, y) \tag{16}$$

We start the problem by applying the Green-Gauss theorem twice to show that

$$\int_\Omega G \nabla^2 u d\Omega = \int_{\partial\Omega} \left(G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n} \right) ds + \int_\Omega u \nabla^2 G d\Omega = \int_\Omega G \phi(x, y) d\Omega. \tag{17}$$

Once again we will choose G so that $G(x, 0) = 0$ in order to get rid of the boundary integral that contains $\frac{\partial u}{\partial n}$. Putting in the definition of the Green's function we have that

$$u(\xi, \eta) = -\int_\Omega G \phi(x, y) d\Omega - \int_{\partial\Omega} \left(u \frac{\partial G}{\partial n} \right) ds. \tag{18}$$

The Green's function for this example is identical to the last example because a Green's function is defined as the solution to the homogenous problem $\nabla^2 u = 0$ and both of these examples have the same homogeneous problem. Putting G into the equation gives

$$u(\xi, \eta) = \frac{\eta}{\pi} \int_{-\infty}^{\infty} \left(\frac{f(x)}{\left[(\xi - x)^2 + \eta^2 \right]} \right) dx + \tag{19}$$

$$\frac{1}{4\pi}\int_0^\infty\int_{-\infty}^\infty \phi(x,y)\ln\left(\frac{(\xi-x)^2+(\eta-y)^2}{(\xi-x)^2+(\eta+y)^2}\right)dxdy \tag{20}$$

4 Example of Laplace's Equation on the Quarter Plane

Finally we will consider Laplace's equation on the 1/4 plane with mixed boundary conditions. The PDE we would like to solve is

$$\nabla^2 u = 0 \tag{21}$$

$$u(x,0) = f(x) \tag{22}$$

$$\frac{\partial u}{\partial n}(0,y) = h(y) \tag{23}$$

$$\tag{24}$$

Just like in the last two examples we will use the Green-Gauss theorem to get

$$\int_\Omega G\nabla^2 u d\Omega = \int_{\partial\Omega}\left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\right)ds + \int_\Omega u\nabla^2 G d\Omega = 0 \tag{25}$$

implementing the boundary conditions for the 1/4 plane gives

$$\int_\Omega G\nabla^2 u d\Omega = \int_\infty^0 \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\right)\bigg|_{x=0} dy + \tag{26}$$

$$\int_0^\infty \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\right)\bigg|_{y=0} dx + \int_\Omega u\nabla^2 G d\Omega = 0 \tag{27}$$

$$\tag{28}$$

where the limits on the integral over y are switched so that we can evaluate the boundary counterclockwise. In this problem we will choose G so that $G(x,0) = 0$ in order to get rid of the boundary integral that contains $\frac{\partial u}{\partial n}$, just like before. But in the integral over y we need to choose G so that $\frac{\partial G}{\partial n} = 0$

on the boundary $x = 0$ because we know the value of the flux across the boundary, but not the value of u. Inserting the Green's function we get

$$u(\xi, \eta) = -\int_0^\infty G h(y)\big|_{x=0} \, dy - \int_0^\infty f(x) \left.\frac{\partial G}{\partial n}\right|_{y=0} dx. \tag{29}$$

Now $G = w + g$ and $w = \frac{1}{2\pi} \ln r$ as before. In the last examples we introduced a negative point source at $(\xi, -\eta)$ in order to enforce $G(x, 0) = 0$. This will also work here:

$$G = \frac{1}{2\pi} \ln \sqrt{(\xi - x)^2 + (\eta + y)^2} - \frac{1}{2\pi} \ln \sqrt{(\xi - x)^2 + (\eta - y)^2} + \ \mathit{bdry\ terms} \tag{30}$$

Since a negative image point source worked to force a constant G boundary, it makes sense that a positive image point source will force $\frac{\partial G}{\partial n} = 0$ on the $x = 0$ boundary. By inspection you can see that we will need two image point sources. One to counteract the point source at (ξ, η) in the domain, and one to counteract the image point source at $(\xi, -\eta)$ that ensures the boundary condition at $y = 0$ is being met. This means that the Green's function for this PDE is

$$\begin{aligned}
G &= \frac{-1}{2\pi} \ln r_1 + \frac{1}{2\pi} \ln r_2 - \frac{1}{2\pi} \ln r_3 + \frac{1}{2\pi} \ln r_4 = \frac{1}{2\pi} \ln \left(\frac{r_2 r_4}{r_1 r_3} \right) && (31)\\
r_1 &= \sqrt{(\xi - x)^2 + (\eta - y)^2} && (32)\\
r_2 &= \sqrt{(\xi - x)^2 + (\eta + y)^2} && (33)\\
r_3 &= \sqrt{(\xi + x)^2 + (\eta - y)^2} && (34)\\
r_4 &= \sqrt{(\xi + x)^2 + (\eta + y)^2} && (35)\\
& && (36)
\end{aligned}$$

and u is given in Eq. 29. In this example we have traded the original PDE for two integrals along the boundaries. These can be integrated analytically as long as $h(y)$ and $f(x)$ are sufficiently simple for the integrals to exist.

5 The Freespace Green's Function for the Diffusion Equation

The diffusion equation is an example of a nonself-adjoint operator. The operator L is given by:

$$L(u) = \frac{\partial u}{\partial t} - \nabla \cdot D \nabla u \tag{37}$$

and an example of a diffusion equation problem with mixed boundary conditions is

$$\begin{aligned} \frac{\partial u}{\partial t} - \nabla \cdot D \nabla u &= F(x,t) && (38) \\ u(a) &= u_a && (39) \\ \frac{\partial u(b)}{\partial t} &= \hat{q}_b. && (40) \end{aligned}$$

In order to simplify the derivation assume that $D = 1$ and $F = 0$ and that we have one-dimensional flow. We need to know the freespace Green's function of the operator L. As before we multiply by the weighting function w and integrate over the domain Ω

$$\begin{aligned} &\int_\Omega \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} w d\Omega = 0 && (41) \\ &\Omega = [0, \infty] \times [-\infty, \infty] && (42) \end{aligned}$$

in this example Ω is a semi-infinite domain in t because negative time is not physically meaningful. Both of the terms in Eq. (42) must be integrated by parts in order to find the adjoint operator of L. This gives

$$\begin{aligned} &\int_\Omega uwd\Omega\bigg|_{t=0}^{T} - \int_t \int_\Omega u \frac{\partial w}{\partial t} d\Omega dt && (43) \\ &- \int_t \int_{\partial\Omega} (\nabla u \cdot n) wdsdt + \int_t \int_{\partial\Omega} (\nabla w \cdot n) udsdt + \int_t \int_\Omega \nabla^2 wud\Omega dt && (44) \end{aligned}$$

$$= \int_{\Omega} u(T,x)\, w(T,x)\, d\Omega - \int_{\Omega} u(0,x)\, w(0,x)\, d\Omega \tag{45}$$

$$+ \int_t \int_{\partial\Omega} (\nabla w \cdot n)\, u - (\nabla u \cdot n)\, w ds dt - \int_t \int_{\Omega} u \left(\frac{\partial w}{\partial t} + \nabla^2 w \right) d\Omega dt \tag{46}$$

$$= bdry\ terms - \int_t \int_{\Omega} u \left(\frac{\partial w}{\partial t} + \nabla^2 w \right) d\Omega dt. \tag{47}$$

Assuming that we can set all of the boundary terms to zero by some combination of intitial and boundary conditions and constraints on w the adjoint operator, L^* of the diffusion operator is

$$- \left(\frac{\partial w}{\partial t} + \nabla^2 w \right). \tag{48}$$

The freespace Green's function is the solution to the negative diffusion equation with a point source

$$- \frac{\partial w}{\partial t} - \nabla^2 w - \delta(\xi - x, \tau - t) = 0 \tag{49}$$

on the two or three-dimensional domain Ω. In this example we have changed sign convention on the delta function from the last example. The reason for this is that the sign convention $L^*(w) + \delta = 0$ is typically used in the derivation of the boundary element method, while the sign convention $L^*(w) - \delta = 0$ is typically used in the method of Green's functions. As with the various conventions used in Fourier transforms, both are "correct." In Green's functions both conventions result in exactly the same answer. (verify this for yourself)

The one-dimensional case of equation (49) can be solved using a Fourier transform on x

$$- \frac{\partial \hat{w}}{\partial t} + s^2 \hat{w} - \delta(\tau - t)\, e^{-is\xi} = 0 \tag{50}$$

where s has been used to denote the transform variable instead of the customary ω in order to avoid confusing w and ω. Equation (50) can be solved using the integrating factor $e^{-is^2 t}$

$$e^{-s^2 t} \hat{w} \Big|_t^{\infty} = e^{-is\xi} \int_t^{\infty} e^{-s^2 t} \delta(\tau - t)\, dt \tag{51}$$

$$
e^{-s^2 t}\hat{w}\Big|_t^{\infty} = \left\{ \begin{array}{cc} 0 & t > \tau \\ e^{-is\xi}e^{-s^2\tau} & t < \tau \end{array} \right. \tag{52}
$$

$$
\hat{w} = e^{-is\xi}e^{-s^2(\tau - t)}H(\tau - t) \tag{53}
$$

this can be inverted to find

$$
w = \frac{H(\tau - t)}{\sqrt{4\pi(\tau - t)}} \exp\left(\frac{-(\xi - x)^2}{4(\tau - t)} \right). \tag{54}
$$

Going back to the original equation we have that

$$
-u(\xi, \tau) = \lim_{T \to \infty} \int_\Omega u(x, T)\, G(x, T)\, d\Omega - \int_\Omega u(x, 0)\, G(x, 0)\, d\Omega \tag{55}
$$

$$
+ \int_t \int_{\partial\Omega} (\nabla G \cdot n)\, u - (\nabla u \cdot n)\, G ds dt \tag{56}
$$

where $G = w + g$ is the Green's function for a particular problem.

As in the Laplace equation example we can choose to solve for the correct Green's function for a specific set of boundary conditions.

6 Example of the Diffusion Equation on the Half-Space $x > 0$

Suppose

$$
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{57}
$$

$$
\Omega = (0 < x < \infty), (0 < t < T) \tag{58}
$$

$$
u(x, 0) = u_o(x) \tag{59}
$$

$$
u(x, T) = u_T(x) \tag{60}
$$

$$
\lim_{x \to \infty} u(x, t) = 0. \tag{61}
$$

The first two boundary terms mean that $u = 0$ on $\partial\Omega$. Since we don't know $\nabla u \cdot n = \frac{du}{dx}$ on the boundary we will make $G = 0$ on $\partial\Omega$. This means that $G(0, t) = 0$ since the domain in semi-infinite. Now

$$-u(\xi,\tau) = \lim_{T\to\infty} \int_\Omega u(x,T)\, G(x,T)\, d\Omega - \int_\Omega u(x,0)\, G(x,0)\, d\Omega \tag{62}$$

where once again $G = w+g$ and g is some function that is zero everywhere in the domain and makes $G(0,t) = 0$. Like the Laplace equation we will solve this by the method of images. In this case we are trying to force $G(x,t) = 0$ on the line $x = 0$ for all $t > 0$. A negative point source at $(-\xi,\tau)$ will work so

$$g = \frac{-H(\tau - t)}{\sqrt{4\pi(\tau - t)}} \exp\left(\frac{-(\xi + x)^2}{4(\tau - t)}\right) \tag{63}$$

and

$$G = w + g = \frac{H(\tau - t)}{\sqrt{4\pi(\tau - t)}} \left[\exp\left(\frac{-(\xi - x)^2}{4(\tau - t)}\right) - \exp\left(\frac{-(\xi + x)^2}{4(\tau - t)}\right)\right]. \tag{64}$$

7 Diffusion Equation on a Bounded Interval

Suppose we want to solve the same equation as in the last example, except that now $\Omega = [0, L]$. and $u(L,t) = 0$. Since we don't know $\frac{du}{dx}$ on the boundary we will make $G = 0$ on $\partial\Omega$. This means that $G(0,t) = 0$ and $G(L,t) = 0$ since the domain in bounded.

Initially we need to introduce a negative point source at $(-\xi,\tau)$ in order to enforce $G(0,t) = 0$. This creates a flux across the boundary at $x = L$. In order to balance this flux, a negative point source at $(2L - \xi, \tau)$ must be introduced. The new point souce now creates flux across the boundary at $G(0,t)$ and must be balanced by a positive point souce at $(-2L + \xi, \tau)$.

We must continue in this way for an infinite number of point sources. The final Green's function of this problem is given by

$$G = \frac{H(\tau - t)}{\sqrt{4\pi(\tau - t)}} \tag{65}$$

$$\left[\sum_{L=-\infty}^{L=\infty} \exp\left(\frac{-\left(2L+\xi-x\right)^2}{4\left(\tau-t\right)}\right) - \sum_{L=-\infty}^{L=\infty} \exp\left(\frac{-\left(2L-\xi-x\right)^2}{4\left(\tau-t\right)}\right)\right]$$

and

$$u\left(\xi,\tau\right) = \int_0^L u_o\left(x\right) G\left(x,0\right) dx - \int_0^L u_T\left(x\right) G\left(x,T\right) dx \qquad (66)$$

8 Summary

In the method of Green's functions we take PDEs that can't be solved by transforms and

- Integrate them by parts (apply the Green-Gauss theorem in 2D) two times so that we are left with *bdry terms* $+ \int_\Omega L^*(G)ud\Omega$
- Find the freespace Green's function w that satisfies $L^*(w) - \delta = 0$.
- Use the method of images to find the Green's function $G = w + g$ for a given set of initial and boundary conditions.
- Plug the function G bac into the integral and integrate to find u at an arbitrary point $(-\xi, \tau)$ or $(-\xi, \eta)$.
- We can find u at any point in the domain this way.

References

[1] Greenberg, M.D. *Foundations of Applied Mathematics*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1978.

[2] Juanes, R. *PE281 Course Notes*, Stanford University, 2003.