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ENERGY 281
Spring Quarter 2007-08
Homework Assignment 1 Solution

1. The following equation governs the vorticity vector in an incompressible viscous fluid with no body forces, but dont worry if that doesnt mean anything to you. Your task is to write the equation in dimensionless form.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) + \frac{\mu}{\rho}\frac{\partial^2 v}{\partial x^2} \tag{1}$$

where $v =$ velocity, $\frac{m}{s}$
$t =$ time, s
$x =$ distance, m
$p =$ pressure, $\frac{kg}{ms^2}$
$\rho =$ density, $\frac{kg}{m^3}$
$\mu =$ viscosity, $\frac{kg}{ms}$

Feel free to introduce any other quantities you may need. There may be several possible solutions to this problem.

Solution Start by introducing characteristic length and velocity scales, L and V. This implies the following choice of dimensionless variables:

$$\begin{aligned} x_D &= \frac{x}{L} \\ v_D &= \frac{v}{V} \\ t_D &= \frac{Vt}{L} \end{aligned}$$

Substitute these into the governing differential equation:

$$\frac{V^2}{L}\frac{\partial v_D}{\partial t_D} + \left(\frac{V^2}{L}\right) v_D \frac{\partial v_D}{\partial x_D} = -\frac{1}{L}\frac{\partial}{\partial x_D}\left(\frac{p}{\rho}\right) + \frac{V}{L^2}\frac{\mu}{\rho}\frac{\partial^2 v_D}{\partial x_D^2}$$

For convenience define a new dimensionless group p_D:

$$p_D = \frac{p}{\rho V^2}$$

Simplify the dimensionless form of the equation:

$$\frac{\partial v_D}{\partial t_D} + v_D\frac{\partial v_D}{\partial x_D} = -\frac{\partial pD}{\partial x_D} + \left(\frac{\mu}{VL\rho}\right)\frac{\partial^2 v_D}{\partial x_D^2}$$

2. Instead of a slightly compressible fluid consider the flow of a real gas in a homogeneous medium. The PVT behavior for this case is given by:

$$pV = ZRT \tag{2}$$

Derive the pressure equation in radial coordinates (only consider r and t) for this case. Note that the viscosity μ is now a function of pressure. Hints: Youll need an expression for the gas density.

$$\rho = \frac{pM}{ZRT} \tag{3}$$

Begin with a material balance, derived in the same way as in the 1-D pressure equation at the beginning of Chapter 1 in Archer's notes, except that this is radial flow, so youll be using a cylindrical control volume of height h, radius r, and thickness Δr.

Also, use the gas compressibility

$$c_g = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T.$$

Solution Begin from a material balance:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho\frac{k}{\mu}\frac{\partial p}{\partial r}\right) = \frac{\partial \rho\phi}{\partial t}$$

Substitute in the real gas law:

$$\rho = \frac{pM}{ZRT}$$

$$c_g = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T = \frac{1}{p} - \frac{1}{Z}\frac{\partial Z}{\partial p}$$

Also use the definition of the rock compressibility:

$$c_r = \frac{1}{\phi}\left(\frac{\partial \phi}{\partial p}\right)_T$$

$$\frac{\partial \phi}{\partial t} = \phi c_r \frac{\partial p}{\partial t}$$

Substitute ρ, c_g, c_r into the material balance equation:

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{p}{\mu Z}\frac{\partial p}{\partial r}\right) &= \frac{p}{Z}\frac{\phi}{k}c_r\frac{\partial p}{\partial t} + \frac{\phi}{k}\frac{\partial\left(\frac{p}{Z}\right)}{\partial t} \\
&= \frac{\phi}{k}\left[\frac{p}{Z}c_r\frac{\partial p}{\partial t} + \frac{1}{Z}\frac{\partial p}{\partial t} - \frac{p}{Z^2}\frac{\partial Z}{\partial t}\right] \\
&= \left(\frac{p}{Z}\right)\left(\frac{\phi}{k}\right)\left[c_r\frac{\partial p}{\partial t} + \frac{1}{p}\frac{\partial p}{\partial t} - \frac{1}{Z}\frac{\partial Z}{\partial p}\frac{\partial p}{\partial t}\right] \\
&= \left(\frac{p}{Z}\right)\left(\frac{\phi}{k}\right)(c_r + c_g)\frac{\partial p}{\partial t}
\end{aligned}$$

This can not be simplified further because the equation is nonlinear.

3. Show that by defining a pseudo-pressure:

$$m(p) = \int_{p_{ref}}^{p} \frac{p}{\mu Z}\, dp$$

where p_{ref} is a some reference pressure, the flow equation for a real gas (derived in problem 2) can be simplified considerably.

Solution

$$m(p) = \int_{p_{ref}}^{p} \frac{p}{\mu Z}\, dp$$

$$\frac{\partial m(p)}{\partial r} = \frac{\partial m(p)}{\partial p}\frac{\partial p}{\partial r} = \frac{p}{\mu Z}\frac{\partial p}{\partial r}$$

$$\frac{\partial m(p)}{\partial t} = \frac{\partial m(p)}{\partial p}\frac{\partial p}{\partial t} = \frac{p}{\mu Z}\frac{\partial p}{\partial t}$$

Substitute into the flow equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial m}{\partial r}\right) = \frac{\phi\mu(c_r + c_g)}{k}\frac{\partial m}{\partial t}$$

4. Let K be a positive constant. Use separation of variables, and Fourier sine and cosine series, to solve the PDE

$$\frac{\partial p}{\partial t} = K\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right), \tag{4}$$

on the rectangle $0 < x < 2$, $0 < y < 1$, with initial condition

$$p(x, y, 0) = 3\cos\pi x \sin 2\pi y - 4\cos 2\pi x \sin 4\pi y, \tag{5}$$

and boundary conditions

$$\frac{\partial p}{\partial x}(0,y,t) = \frac{\partial p}{\partial x}(2,y,t) = 0, \quad p(x,0,t) = p(x,1,t) = 0. \tag{6}$$

Solution Assume that

$$p(x,y,t) = P(x)Q(y)N(t).$$

Substituting this form into the PDE, we obtain

$$P(x)Q(y)N'(t) = K(P''(x)Q(y)N(t) + P(x)Q''(y)N(t)).$$

Dividing through by $KP(x)Q(y)N(t)$ yields

$$\frac{N'(t)}{KN(t)} = \frac{P''(x)}{P(x)} + \frac{Q''(y)}{Q(y)}.$$

Because the first term depends on t, the second term depends on x, and the third term depends on y, all three must be equal to constants. Assume that $P''(x)/P(x) = -\lambda$, and $Q''(y)/Q(y) = -\mu$, for some constants λ and μ. Then, we have the ODEs

$$P''(x) + \lambda P(x) = 0, \tag{7}$$

$$Q''(y) + \mu Q(y) = 0, \tag{8}$$

$$N'(t) + K(\lambda + \mu)N(t) = 0, \tag{9}$$

where the constants λ and μ must be determined.

We now solve

$$P''(x) + \lambda P(x) = 0, \quad P'(0) = P'(2) = 0.$$

The general solution is

$$P(x) = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x.$$

To satisfy the boundary conditions, we must have $A = 0$ and $2\sqrt{\lambda} = j\pi$ for some integer j. Therefore $\lambda = (j\pi/2)^2$. The integer j must be nonnegative, because $j = 0$ leads to $P(x) \equiv B$, but $j < 0$ leads to duplicate eigenfunctions.

We then have

$$P(x) = B\cos(j\pi x/2), \quad j = 0, 1, 2, \ldots.$$

We set the constant $B = 1$. Similarly, $\mu = (k\pi)^2$, where k is a positive integer, so

$$Q(y) = \sin(k\pi y), \quad k = 1, 2, \ldots.$$

Now, we solve

$$N'(t) + K(\lambda + \mu)N(t).$$

The general solution is $N(t) = ae^{-K(\lambda+\mu)t}$, where a is a constant.

We assume that the solution $p(x, y, t)$ of the original problem has the form

$$p(x, y, t) = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} p_{jk}(x, y, t),$$

where each term

$$p_{jk}(x, y, t) = P_j(x)Q_k(y)N_{jk}(t) = a_{jk}e^{-K(\lambda_j+\mu_k)t} \cos(j\pi x/2) \sin(k\pi y)$$

satisfies the PDE and boundary conditions. We must now ensure that it satisfies the initial condition. Substituting $t = 0$ yields

$$p(x, y, 0) = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} a_{jk}P_j(x)Q_k(y).$$

The constants a_{jk} must be chosen so that the initial condition is satisfied. Because the functions $P_j(x)$ and $Q_k(y)$ are orthogonal, we see that taking the inner product of both sides of the initial condition

$$p(x, y, 0) = 3 \cos \pi x \sin 2\pi y - 4 \cos 2\pi x \sin 4\pi y$$

with $P_m(x)Q_n(y) = \cos(m\pi x/2) \sin(n\pi y)$, for integers m and n, yields

$$a_{mn} = \frac{(\cos(m\pi x/2) \sin(n\pi y), 3 \cos \pi x \sin 2\pi y - 4 \cos 2\pi x \sin 4\pi y)}{(\cos(m\pi x/2) \sin(n\pi y), \cos(m\pi x/2) \sin(n\pi y))}.$$

It follows from the identities

$$\int_0^L \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & m \neq n, \\ L/2 & m = n, m \neq 0 \end{cases}$$

$$\int_0^L \sin\left(\frac{m\pi y}{L}\right) \sin\left(\frac{n\pi y}{L}\right) dy = \begin{cases} 0 & m \neq n, \\ L/2 & m = n, m \neq 0 \end{cases}$$

that

$$a_{2,2} = 3, \quad a_{4,4} = -4,$$

and all other constants a_{mn} are zero. We conclude that

$$p(x, y, t) = 3e^{-5K\pi^2 t} \cos(\pi x) \sin(2\pi y) - 4e^{-20K\pi^2 t} \cos(2\pi x) \sin(4\pi y).$$