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ENERGY 281
Spring Quarter 2007-08
Homework Assignment 2

This assignment is due on Thursday, April 24. You *must* show your work to receive full credit.

1. Use the full Fourier transform (not a sine or cosine transform) to solve Laplace's equation in the upper half-plane,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad y > 0, \tag{1}$$

with the boundary conditions

$$u(x,0) = f(x), \quad -\infty < x < \infty \tag{2}$$

$$\lim_{|x|\to\infty, y\to\infty} u(x,y) = 0. \tag{3}$$

For full credit, your final solution must involve only a single integral, and no complex numbers.

2. Review Example 3 (Laplace's equation on a semi-infinite strip) in the Lecture 4 notes. Use the fact that for $as \gg 1$,

$$\frac{\sinh[s(a-y)]}{\sinh(sa)} \approx e^{-sy} \tag{4}$$

to show that as $a \to \infty$, the solution to this problem approaches the solution to Laplace's equation in the quarter plane,

$$p(x,y) = \frac{y}{\pi}\int_0^\infty \left\{ \frac{1}{(x-\lambda)^2 + y^2} - \frac{1}{(x+\lambda)^2 + y^2} \right\} f(\lambda)\, d\lambda. \tag{5}$$

You'll need to use a trigonometric identity that has been previously used in lecture.

3. Solve the heat equation in a semi-infinite interval:

$$\frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < \infty, \quad t > 0, \tag{6}$$

$$u(x,0) = u_0, \quad 0 < x < \infty, \tag{7}$$

$$u(0,t) = 0, \quad t > 0. \tag{8}$$

For full credit, your final answer must *not* involve any integrals.

4. In this problem, we work with the complex form of the Fourier series for a function defined on the interval $[0, L]$:

$$f(x) = \frac{1}{\sqrt{L}} \sum_{\omega=-\infty}^{\infty} \hat{f}(\omega) e^{2\pi i \omega x / L}$$

where the coefficients $\{\hat{f}(\omega)\}_{\omega=-\infty}^{\infty}$ are given by

$$\hat{f}(\omega) = \frac{1}{\sqrt{L}} \int_0^L f(x) e^{-2\pi i \omega x / L} \, dx.$$

(a) Compute the coefficients of the Fourier series of $f(x) = (x - \pi)^2$ on the interval $[0, 2\pi]$.

(b) Compute the coefficients of the Fourier series of

$$g(x) = \begin{cases} 1/2 & 0 \le x < \pi \\ -1/2 & \pi \le x < 2\pi \end{cases} .$$

(c) A function $f(x)$ defined on an interval I (which can be the entire real line) is said to be *square-integrable* if

$$\int_I |f(x)|^2 \, dx$$

is finite. Let $I = [0, 2\pi]$. Prove that the coefficients $\{\hat{f}(\omega)\}$ of the Fourier series of a real-valued square-integrable function $f(x)$ defined on I must decay to zero as $|\omega| \to \infty$.

(d) Which of the functions from parts 4a and 4b have Fourier coefficients that decay to zero more rapidly as $|\omega| \to \infty$? Why do you think that is the case?