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**ENERGY 281**
**Spring Quarter 2007-08**
**Homework Assignment 2 Solution**

1. Use the full Fourier transform (not a sine or cosine transform) to solve Laplace's equation in the upper half-plane,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad y > 0, \tag{1}$$

with the boundary conditions

$$u(x,0) = f(x), \quad -\infty < x < \infty \tag{2}$$

$$\lim_{|x|\to\infty, y\to\infty} u(x,y) = 0. \tag{3}$$

For full credit, your final solution must involve only a single integral, and no complex numbers.

**Solution** The transformed equation is the ODE

$$\frac{\partial^2 \hat{u}}{\partial y^2} - s^2 \hat{u} = 0,$$

with "initial" condition

$$\hat{u}(s,0) = \hat{f}(s).$$

This ODE has the solution

$$\hat{u}(s,y) = \hat{f}(s)e^{-|s|y}.$$

Therefore, the solution is

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \hat{u}(s,y)\, ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx-|s|y} \hat{f}(s)\, ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{is(x-z)-|s|y} f(z)\, dz\, ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z) \left[ \int_{-\infty}^{\infty} e^{is(x-z)-|s|y}\, ds \right] dz \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)\left[\int_{-\infty}^{0} e^{is(x-z)+sy}\,ds\right]\,dz + \\
&\quad \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)\left[\int_{0}^{\infty} e^{is(x-z)-sy}\,ds\right]\,dz \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)\left[\frac{1}{y+i(x-z)}\right]\,dz + \\
&\quad \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)\left[\frac{1}{y-i(x-z)}\right]\,dz \\
&= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(z)}{(x-z)^2+y^2}\,dz
\end{aligned}
$$

2. Review Example 3 (Laplace's equation on a semi-infinite strip) in the Lecture 4 notes. Use the fact that for $as \gg 1$,

$$\frac{\sinh[s(a-y)]}{\sinh(sa)} \approx e^{-sy} \tag{4}$$

to show that as $a \to \infty$, the solution to this problem approaches the solution to Laplace's equation in the quarter plane,

$$p(x,y) = \frac{y}{\pi}\int_0^{\infty}\left\{\frac{1}{(x-\lambda)^2+y^2} - \frac{1}{(x+\lambda)^2+y^2}\right\} f(\lambda)\,d\lambda. \tag{5}$$

You'll need to use a trigonometric identity that has been previously used in lecture.

**Solution** From the solution in the Lecture 4 notes, we have

$$p(x,y) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(\lambda)e^{-sy}\sin(s\lambda)\sin(sx)\,d\lambda\,ds.$$

Using the product to sum identity

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

we obtain

$$
\begin{aligned}
p(x,y) &= \frac{1}{\pi}\int_0^{\infty} f(\lambda)\int_0^{\infty} e^{-sy}\cos[s(x-\lambda)] - \cos[s(x+\lambda)]\,ds\,d\lambda \\
&= \frac{1}{\pi}\int_0^{\infty} f(\lambda)\left[\frac{y}{y^2+(x-\lambda)^2}\right]\,d\lambda - \\
&\quad \frac{1}{\pi}\int_0^{\infty} f(\lambda)\left[\frac{y}{y^2+(x+\lambda)^2}\right]\,d\lambda \\
&= \frac{y}{\pi}\int_0^{\infty}\left\{\frac{1}{(x-\lambda)^2+y^2} - \frac{1}{(x+\lambda)^2+y^2}\right\} f(\lambda)\,d\lambda.
\end{aligned}
$$

3. Solve the heat equation in a semi-infinite interval:

$$\frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < \infty, \quad t > 0, \tag{6}$$

$$u(x,0) = u_0, \quad 0 < x < \infty, \tag{7}$$

$$u(0,t) = 0, \quad t > 0. \tag{8}$$

For full credit, your final answer must *not* involve any integrals.

**Solution** Using the Fourier sine transform yields the ODE

$$\frac{\partial \hat{u}}{\partial t} + s^2 c^2 \hat{u} = 0$$

with initial condition

$$\hat{u}(s,0) = \mathcal{F}(u_0) = \hat{u}_0(s).$$

The solution to this ODE is

$$\hat{u}(s,t) = \hat{u}_0(s) e^{-s^2 c^2 t}.$$

Inverting the transform yields

$$\begin{aligned}
u(x,t) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}(s,t) \sin(sx)\, ds \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}_0(s) e^{-s^2c^2t} \sin(sx)\, ds \\
&= \frac{2u_0}{\pi} \int_0^\infty \int_0^\infty e^{-s^2c^2t} \sin(sy) \sin(sx)\, dy\, ds \\
&= \frac{u_0}{\pi} \int_0^\infty \int_0^\infty e^{-s^2c^2t} \cos[s(y-x)]\, dy\, ds - \\
&\quad \frac{u_0}{\pi} \int_0^\infty \int_0^\infty e^{-s^2c^2t} \cos[s(y+x)]\, dy\, ds \\
&= \frac{u_0}{2\pi} \int_0^\infty \int_0^\infty e^{-s^2c^2t+is(y-x)}\, dy\, ds + \\
&\quad \frac{u_0}{2\pi} \int_0^\infty \int_0^\infty e^{-s^2c^2t-is(y-x)}\, dy\, ds - \\
&\quad \frac{u_0}{2\pi} \int_0^\infty \int_0^\infty e^{-s^2c^2t+is(y+x)}\, dy\, ds - \\
&\quad \frac{u_0}{2\pi} \int_0^\infty \int_0^\infty e^{-s^2c^2t-is(y+x)}\, dy\, ds
\end{aligned}$$

$$\begin{aligned}
&= \frac{u0}{2\pi}\int_0^\infty \int_{-\infty}^\infty e^{-s^2c^2t+is(y-x)}\,ds\,dy - \\
&\quad \frac{u_0}{2\pi}\int_0^\infty \int_{-\infty}^\infty e^{-s^2c^2t+is(y+x)}\,ds\,dy \\
&= \frac{u_0}{\sqrt{4\pi c^2 t}}\int_0^\infty e^{-(y-x)^2/(4c^2t)}\,dy - \\
&\quad \frac{u_0}{\sqrt{4\pi c^2 t}}\int_0^\infty e^{-(y+x)^2/(4c^2t)}\,dy \\
&= \frac{u_0}{\sqrt{\pi}}\int_{-x/\sqrt{4c^2t}}^\infty e^{-y^2}\,dy + \frac{u_0}{\sqrt{\pi}}\int_\infty^{x/\sqrt{4c^2t}} e^{-y^2}\,dy \\
&= \frac{u_0}{\sqrt{\pi}}\int_{-x/\sqrt{4c^2t}}^{x/\sqrt{4c^2t}} e^{-y^2}\,dy \\
&= \frac{2u_0}{\sqrt{\pi}}\int_0^{x/\sqrt{4c^2t}} e^{-y^2}\,dy \\
&= u_0\text{erf}\left(\frac{x}{\sqrt{4c^2t}}\right).
\end{aligned}$$

4. In this problem, we work with the complex form of the Fourier series for a function defined on the interval $[0, L]$:

$$f(x) = \frac{1}{\sqrt{L}}\sum_{\omega=-\infty}^{\infty} \hat{f}(\omega)e^{2\pi i\omega x/L}$$

where the coefficients $\{\hat{f}(\omega)\}_{\omega=-\infty}^{\infty}$ are given by

$$\hat{f}(\omega) = \frac{1}{\sqrt{L}}\int_0^L f(x)e^{-2\pi i\omega x/L}\,dx.$$

   (a) Compute the coefficients of the Fourier series of $f(x) = (x-\pi)^2$ on the interval $[0, 2\pi]$.

   **Solution** Using integration by parts, we obtain

$$\hat{f}(\omega) = \sqrt{2\pi}\frac{2}{\omega^2}, \quad \omega \neq 0, \quad \hat{f}(0) = \sqrt{2\pi}\frac{\pi^2}{3}.$$

   (b) Compute the coefficients of the Fourier series of

$$g(x) = \begin{cases} 1/2 & 0 \leq x < \pi \\ -1/2 & \pi \leq x < 2\pi \end{cases}.$$

**Solution** By breaking up the integral from 0 to $2\pi$ into two integrals, one from 0 to $\pi$ and one from $\pi$ to $2\pi$, we obtain

$$\hat{f}(\omega) = \frac{i[(-1)^\omega - 1]}{\sqrt{2\pi}\omega}, \quad \omega \neq 0, \quad \hat{f}(0) = 0.$$

(c) A function $f(x)$ defined on an interval $I$ (which can be the entire real line) is said to be *square-integrable* if

$$\int_I |f(x)|^2 \, dx$$

is finite. Let $I = [0, 2\pi]$. Prove that the coefficients $\{\hat{f}(\omega)\}$ of the Fourier series of a real-valued square-integrable function $f(x)$ defined on $I$ must decay to zero as $|\omega| \to \infty$.

**Solution** By Parsevals identity,

$$\sum_{\omega=-\infty}^{\infty} |\hat{f}(\omega)|^2 = \int_0^{2\pi} |f(x)|^2 \, dx$$

is finite. Since $f$ is real, $\hat{f}(-\omega) = \hat{f}(\omega)$, it follows that the sums

$$\sum_{\omega=1}^{\infty} |\hat{f}(\omega)|^2, \quad \sum_{\omega=1}^{\infty} |\hat{f}(-\omega)|^2$$

are both finite (in fact, they are equal). Because each series has an infinite number of terms but has a finite sum, the terms of each series must approach 0 as $\omega \to \infty$. Therefore, $|\hat{f}(\omega)| \to 0$ as $|\omega| \to \infty$.

(d) Which of the functions from parts 4a and 4b have Fourier coefficients that decay to zero more rapidly as $|\omega| \to \infty$? Why do you think that is the case?

**Solution** The coefficients of $f(x) = (x - \pi)^2$ decay like $|\omega|^{-2}$, while those of $g(x)$ decay like $|\omega|^{-1}$. This is because $f(x)$, being a function whose periodic extension is continuous, is more readily approximated by a smaller set of smooth functions such as those of the form $e^{i\omega x}$.