Jim Lambers
ENERGY 281
Spring Quarter 2007-08
Homework Assignment 3 Solution

1. Plot the exponential integral solution

$$p_D(r_D, t_D) = \frac{1}{2}Ei\left(-\frac{r_D^2}{4t_D}\right),$$

and the late time approximation to this solution,

$$p_D(r_D, t_D) = -\frac{1}{2}\left(\ln\frac{t_D}{r_D^2} + 0.80907\right),$$

on the same axes for a range of t_D values and $r_D = 1$. It is recommended that you use values of t_D that vary over several orders of magnitude.

For what values of t_D does the late time approximation closely approximate the exponential integral solution? What real time values does this correspond to?

In field units, t_D is defined as

$$t_D = \frac{0.000264kt}{\phi\mu c_t r_w^2},$$

where t is in hours. Choose

$$\begin{aligned} k &= 200 \text{ md}, \\ \phi &= 0.3, \\ \mu &= 1.0 \text{ cp}, \\ c_t &= 10^{-6} \text{ psi}^{-1}, \\ r_w &= 0.3 \text{ ft}. \end{aligned}$$

The quickest, easiest way to do this is in a tool such as Matlab, Mathematica or Maple.

2. Consider a linear one-dimensional reservoir. The inner boundary condition is constant pressure (p_w) with skin (S). The outer boundary

condition is infinite acting. The initial-boundary value problem to be solved is

$$\frac{\partial^2 p}{\partial x^2} = \frac{\phi \mu c_t}{k} \frac{\partial p}{\partial t},$$

with initial condition

$$p(x, 0) = p_i,$$

inner boundary condition

$$p(0, t) - S \frac{\partial p}{\partial x}\bigg|_{x=0} = p_w,$$

and condition at infinity

$$\lim_{x \to \infty} p(x, t) = p_i.$$

(a) First, write the problem in dimensionless form. Use the same transformations for x and t as in the Lecture 1 notes. For p, use the same transformation as in the case of a constant pressure boundary condition. Non-dimensionalize the skin using the transformation $S_D = S/L$, where L is the length scale of the reservoir.

(b) Next, solve the dimensionless differential equation in Laplace space.

(c) Finally, invert the transform. Use either a table, or the Maple function `invlaplace`. If you use a table, indicate which table. If you use Maple, submit your code and output.

Solution

$$p_w = p(x = 0) - S \frac{\partial p}{\partial x}|_{x=0^+}$$

Choose a dimensionless pressure and length:

$$p_D = \frac{p_i - p}{p_i - p_w}$$

$$x_D = \frac{x}{L}$$

Also nondimensionalise the skin:

$$S_D = \frac{S}{L}$$

$$\Rightarrow p_w = -p_D(p_i - p_w) + p_i + S_D L \frac{\partial p_D}{\partial x_D} \frac{p_i - p_w}{L}$$
$$\Rightarrow \frac{p_w - p_i}{p_i - p_w} = -p_D + S_D \frac{\partial p_D}{\partial x_D}.$$

Simplifying this gives

$$p_D - S_D \frac{\partial p_D}{\partial x_D} = 1.$$

The governing differential equation is

$$\frac{\partial^2 p_D}{\partial t_D^2} = \frac{\partial p_D}{\partial t_D}.$$

The infinite acting condition is

$$p(\infty, t) = p_i \Rightarrow p_D(\infty, t_D) = 0 \Rightarrow \hat{p}_D(\infty, t_D) = 0.$$

The inner boundary condition is

$$p_D - S_D \frac{\partial p_D}{\partial x_D}|_{x_D=0} = 1 \Rightarrow \hat{p}_D - S_D \frac{\partial \hat{p}_D}{\partial x_D}|_{x_D=0} = \frac{1}{s}.$$

The initial condition is

$$p(x, 0) = p_i \Rightarrow p_D(x_D, 0) = 0.$$

When the differential equation is transformed it becomes

$$\frac{\partial^2 \hat{p}_D}{\partial x_D^2} = s\hat{p}_D.$$

The general solution is

$$\hat{p}_D = c_1 e^{-\sqrt{s} x_D} + c_2 e^{\sqrt{s} x_D}.$$

To ensure this solution remains bounded as $x_D \to \infty$, the constant c_2 is set to zero. Now use the inner condition to find c_1:

$$c_1 + c_1 \sqrt{s} S_D = \frac{1}{s}$$
$$\Rightarrow c_1 = \frac{1}{s(1 + \sqrt{s} S_D)}$$

$$\Rightarrow \hat{p}_D = \frac{e^{-\sqrt{s}x_D}}{s^{3/2}S_D + s}.$$

The transform can be inverted by using the Maple function `invlaplace`:

$$p_D = -e^{\frac{x_D}{S_D}} e^{\frac{t_D}{S_D^2}} \operatorname{erfc}\left(\frac{\sqrt{t_D}}{S_D} + \frac{x_D}{2\sqrt{t_D}}\right) + \operatorname{erfc}\left(\frac{x_D}{2\sqrt{t_D}}\right).$$

3. Solve the Cauchy problem

$$\frac{\partial^2 T}{\partial^2 x} = \alpha \frac{\partial T}{\partial t}, \quad -\infty < x < \infty, \quad t > 0,$$

where α is a constant, with initial condition

$$T(x, 0) = e^{-\pi x^2}, \quad -\infty < x < \infty.$$

Use *both* the Fourier transform and Laplace transform *together* to obtain an *algebraic* equation that can easily be solved, and then invert both transforms to obtain $T(x,t)$. Be careful with your notation to keep track of your transforms.

4. Consider the following sampled signal $g(t)$ on the interval $[0, 1)$:

$$g(t) = \begin{cases} 1 & 0 \le t < 0.25 \\ 0.5 & 0.25 \le t < 0.5 \\ 0.75 & 0.5 \le t < 0.75 \\ 0.25 & 0.75 \le t < 1 \end{cases}.$$

This signal is shown in Figure 1.

(a) This signal can be expressed as a sum of Haar smoothing functions at a particular scale M,

$$g(t) = \sum_{n=-\infty}^{\infty} \phi_{M,n}(t) s_{M,n}.$$

provided that this scale is sufficiently fine. What is the largest value of M for which this is possible? In other words, what is the largest M such that $g \in V_M$? Recall that larger values of M correspond to coarser scales.

Solution Because the signal is piecewise constant on intervals of width $1/4 = 2^{-2}$, it is a linear combination of scaling functions of the form

$$\phi_{-2,n}(t) = \phi\left(\frac{t - n2^{-2}}{2^{-2}}\right)$$

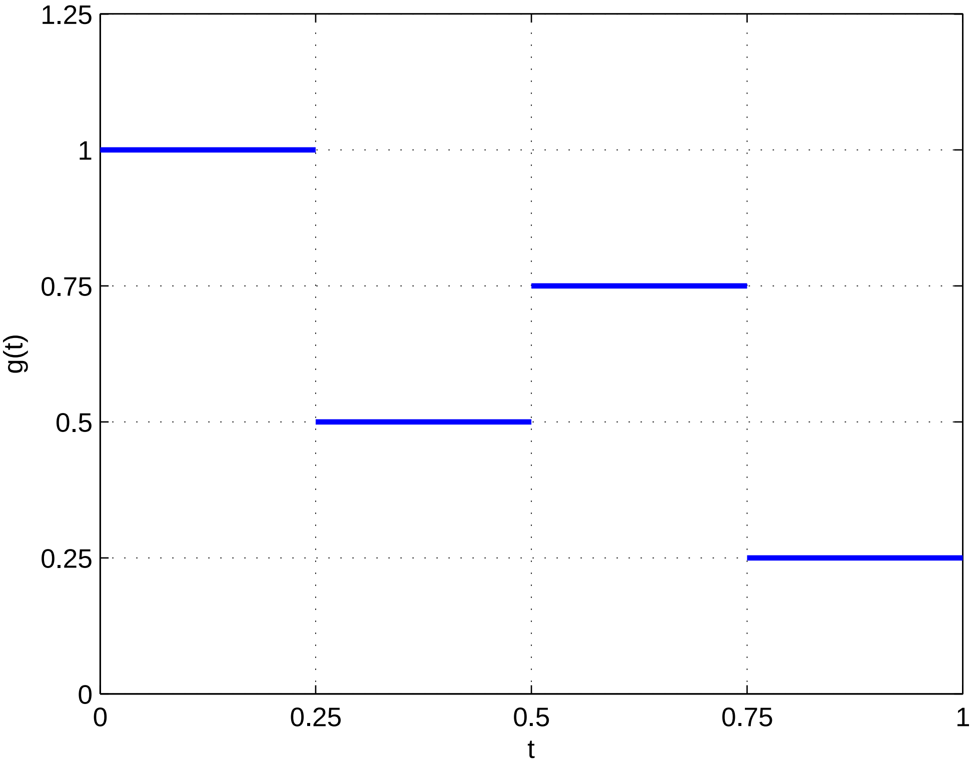

Figure 1: Signal for Problem 4

that are also piecewise constant on intervals of width 2^{-2}, so $M = -2$.

(b) Perform the Haar decomposition of g to compute the detail coefficients $\{d_{M+1,n}\}$, $\{d_{M+2,n}\}$, and $\{s_{M+2,n}\}$, where M is the scale determined in part (a).

Solution Because the signal is supported only on $[0, 1)$, and

$$d_{m,n} = 2^{-m/2} \left[\int_{n2^m}^{(n+1/2)2^m} g(t)\, dt - \int_{(n+1/2)2^m}^{(n+1)2^m} g(t)\, dt \right],$$

it follows that for $m = -1$, we only need to compute coefficients for $n = 0$ and $n = 1$. We have

$$d_{-1,0} = \sqrt{2}\left[\int_0^{1/4} g(t) - \int_{1/4}^{1/2} g(t)\,dt\right] = \sqrt{2}\frac{1}{4}(1 - 0.5) = \frac{\sqrt{2}}{8},$$

$$d_{-1,1} = \sqrt{2}\left[\int_{1/2}^{3/4} g(t) - \int_{3/4}^{1} g(t)\,dt\right] = \sqrt{2}\frac{1}{4}(0.75 - 0.25) = \frac{\sqrt{2}}{8}.$$

Using the formula

$$s_{m,n} = 2^{-m/2}\int_{n2^m}^{(n+1)2^m} g(t)\,dt,$$

we also obtain

$$s_{-1,0} = \sqrt{2}\int_0^{1/2} g(t)\,dt = \sqrt{2}\frac{1}{4}(1 + 0.5) = \frac{3\sqrt{2}}{8},$$

$$s_{-1,1} = \sqrt{2}\int_{1/2}^{1} g(t)\,dt = \sqrt{2}\frac{1}{4}(0.75 + 0.25) = \frac{\sqrt{2}}{4},$$

which yields a smoothed signal belonging to V_{-1},

$$g_{-1}(t) = s_{-1,0}\phi_{-1,0}(t) + s_{-1,1}\phi_{-1,1}(2) = \begin{cases} 3/4 & 0 \leq x < 1/2 \\ 1/2 & 1/2 \leq x < 1 \end{cases}.$$

We can work with this simpler signal to complete the decomposition. We have

$$d_{0,0} = \int_0^{1/2} g_{-1}(t)\,dt - \int_{1/2}^{1} g_{-1}(t)\,dt = \frac{1}{2}(3/4 - 1/2) = \frac{1}{8},$$

$$s_{0,0} = \int_0^{1} g_{-1}(t)\,dt = \frac{1}{2}(3/4 + 1/2) = \frac{5}{8}.$$

(c) Compress the signal by setting to zero the detail coefficients which satisfy

$$|d_{m,n}| \leq 0.5.$$

Reconstruct the compressed signal using the relation

$$\sum_{n=-\infty}^{\infty} s_{m,n}\phi_{m,n}(t) = \sum_{n=-\infty}^{\infty} s_{m+1,n}\phi_{m+1,n}(t) + \sum_{n=-\infty}^{\infty} d_{m+1,n}\psi_{m+1,n}(t).$$

Solution All of the detail coefficients have magnitude less than 0.5, so all of them are dropped, leaving only the approximation coefficient $s_{0,0} = 5/8$. It follows that the reconstructed signal is

$$\tilde{g}(t) = g_0(t) = s_{0,0}\phi_{0,0}(t) = \begin{cases} 5/8 & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

The relation given above shows that because no detail is added to this signal, the compressed signal $\tilde{g}(t)$ at the finest scale of -2 is equal to the signal at scale 0, which is $g_0(t)$.

(d) If $\tilde{g}$ is a compression of a signal g, then the compression ratio is given by

$$100\frac{\|\tilde{g}\|_{L_2}^2}{\|g\|_{L_2}^2} = 100\frac{\sum_{m,n=-\infty}^{\infty} \tilde{d}_{m,n}^2}{\sum_{m,n=-\infty}^{\infty} d_{m,n}^2},$$

where $\{\tilde{d}_{m,n}\}$ is the set of detail coefficients for the compressed signal. Compute and interpret the compression ratio for the compression performed in part (c). *Hint:* $\tilde{d}_{M+3,0} = d_{M+3,0} = \sqrt{2}s_{M+2,0}$.

Solution We have $\tilde{d}_{1,0} = d_{1,0} = 5\sqrt{2}/8$. It follows that the compression ratio S is

$$\begin{aligned} S &= 100\frac{\sum_{m,n=-\infty}^{\infty} \tilde{d}_{m,n}^2}{\sum_{m,n=-\infty}^{\infty} d_{m,n}^2} \\ &= 100\frac{(5\sqrt{2}/8)^2}{(5\sqrt{2}/8)^2 + (1/8)^2 + (\sqrt{2}/8)^2 + (\sqrt{2}/8)^2} \\ &= 90.9091. \end{aligned}$$

The interpretation of this ratio is that a little more than 9% of the data in the signal is lost in the compression. Typical compression algorithms should preserve close to 100% of the data, so the threshold from part (c) should be lowered.