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ENERGY 281
Spring Quarter 2007-08
Lecture 1 Notes

These notes are based on Rosalind Archer's PE281 lecture notes, with some revisions by Jim Lambers.

1 The Diffusion Equation

This course considers slightly compressible fluid flow in porous media. The differential equation governing the flow can be derived by performing a mass balance on the fluid within a control volume.

1.1 One-dimensional Case

First consider a one-dimensional case as shown in Figure 1:

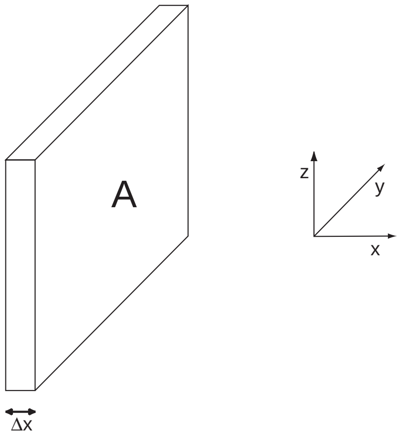

Figure 1: One-dimensional control volume

$$(mass\ \ in) - (mass\ \ out) = (mass\ \ accumulation) \tag{1}$$

$$\Rightarrow \Delta t q\rho|_x - \Delta t q\rho|_{x+\Delta x} = \phi V\rho|_{t+\Delta t} - \phi V\rho|_t \tag{2}$$

where $V = \Delta x A$ and $q = -\frac{kA}{\mu}\frac{\partial p}{\partial x}$.

Dividing (2) through by Δx and Δt and taking limits as $\Delta x \to 0$ and $\Delta t \to 0$ gives:

$$\lim_{\Delta x \to 0} \frac{q\rho|_x - q\rho|_{x+\Delta x}}{\Delta x} = \lim_{\Delta t \to o} \frac{\phi A\rho|_{t+\Delta t} - \phi A\rho|_t}{\Delta t} \tag{3}$$

$$\Rightarrow -\frac{\partial}{\partial x}(q\rho) = \frac{\partial}{\partial t}(\phi A\rho) \tag{4}$$

Substituting Darcy's law into (4) gives:

$$\frac{\partial}{\partial x}\left(\frac{kA}{\mu}\rho\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial t}(\phi A\rho) \tag{5}$$

This equation is sometimes referred to as a *mass balance* or a *material balance.*

Now assume (for simplicity) that k, μ and A are constant:

$$\Rightarrow \frac{\partial}{\partial x}\left(\rho\frac{\partial p}{\partial x}\right) = \frac{\mu}{k}\frac{\partial \phi\rho}{\partial t} \tag{6}$$

Now account for the dependence of ρ on pressure by introducing the isothermal compressibility:

$$c_f = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T \tag{7}$$

where T denotes that the derivative is taken at constant temperature.

Now apply the product rule for differentiation to (6):

$$\frac{\partial}{\partial x}\left(\rho\frac{\partial p}{\partial x}\right) = \frac{\mu}{k}\left(\rho\frac{\partial \phi}{\partial t} + \phi\frac{\partial \rho}{\partial t}\right) \tag{8}$$

The right hand side terms in equation (8) require further attention. First consider the final term, $\phi\frac{\partial\rho}{\partial t}$. By the Chain Rule and (7),

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial p}\frac{\partial p}{\partial t} = c_f\rho\frac{\partial p}{\partial t} \tag{9}$$

Now consider $\frac{\partial\phi}{\partial t}$. First define the rock compressibility as:

$$c_r = \frac{1}{\phi}\left(\frac{\partial \phi}{\partial p}\right)_T \tag{10}$$

$$\Rightarrow \frac{\partial \phi}{\partial t} = \phi c_r\frac{\partial p}{\partial t} \tag{11}$$

Substitute equation (9) and (11) into (8):

$$\frac{\partial}{\partial x}\left(\rho\frac{\partial p}{\partial x}\right) = \frac{\mu}{k}\rho\phi(c_r + c_f)\frac{\partial p}{\partial t} \tag{12}$$

Let $c_t = c_r + c_f$. Now expand the spatial derivative in equation (12):

$$\rho\frac{\partial^2 p}{\partial x^2} + \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial x} = \frac{\phi\mu c_t}{k}\rho\frac{\partial p}{\partial t} \tag{13}$$

Now consider the second term in equation (13):

$$\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial x} = \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial p}\frac{\partial p}{\partial x} = \frac{\partial \rho}{\partial p}\left(\frac{\partial p}{\partial x}\right)^2 = c_f \rho \left(\frac{\partial p}{\partial x}\right)^2 \tag{14}$$

This term is expected to be small, so it is usually neglected. Finally we have:

$$\frac{\partial^2 p}{\partial x^2} = \frac{\phi \mu c_t}{k}\frac{\partial p}{\partial t} \tag{15}$$

2 Three-dimensional Case

The diffusion equation can be expressed using the notation of vector calculus for a general coordinate system as:

$$\nabla^2 p = \frac{\phi \mu c_t}{k}\frac{\partial p}{\partial t} \tag{16}$$

For the case of the radial coordinates the diffusion equation is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2} = \frac{\phi \mu c_t}{k}\frac{\partial p}{\partial t} \tag{17}$$

3 Dimensionless Form

3.1 One Dimensional Problem

The pressure equation for one dimensional flow (equation (15)) can be written in dimensionless form by choosing the following dimensionless variables:

$$p_D = \frac{p_i - p}{p_i}, \quad x_D = \frac{x}{L}, \quad t_D = \frac{kt}{\phi \mu c_t L^2}, \tag{18}$$

where L is a length scale in the problem.

With this choice of dimensionless variables the flow equation becomes:

$$\frac{\partial^2 p_D}{\partial x_D^2} = \frac{\partial p_D}{\partial t_D} \tag{19}$$

This can be seen by applying the Chain Rule to obtain

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_D}\frac{\partial x_D}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x_D}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t_D}\frac{\partial t_D}{\partial t} = \frac{k}{\phi \mu c_t L^2}\frac{\partial}{\partial t_D}. \tag{20}$$

3.2 Radial Problem

The radial form of the pressure equation is usually written in nondimensional form taking account of the boundary conditions. When only radial variations of pressure are considered the pressure equation is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = \frac{\phi\mu c_t}{k}\frac{\partial p}{\partial t} \tag{21}$$

Working with cylindrical control volumes, we obtain the boundary and initial conditions

$$q = \frac{2\pi kh}{\mu} r\frac{\partial p}{\partial r}, \qquad r = r_w \tag{22}$$

$$p = p_i, \qquad r \to \infty, \forall t \tag{23}$$

$$p = p_i, \qquad t = 0, \forall r \tag{24}$$

To nondimensionalize the initial and boundary conditions, we set

$$p_D = \alpha(p_i - p) \tag{25}$$

where α must still be determined. The infinite acting-boundary condition becomes

$$p_D = \alpha(p_i - p_i) = 0, \qquad r \to \infty, \forall t \tag{26}$$

The initial condition becomes

$$p_D = \alpha(p_i - p_i) = 0, \qquad t = 0, \forall r \tag{27}$$

We set the dimensionless length, r_D, and dimensionless time, t_D, to

$$r_D = \frac{r}{r_w}, \quad t_D = \frac{t}{t^*}, \tag{28}$$

where t^* must still be determined.

We substitute p_D and r_D into the pressure equation and obtain

$$\frac{1}{r_D r_w}\frac{\partial}{\partial r_D r_w}\left(r_D r_w \frac{\partial}{\partial r_D r_w}\left(-\frac{p_D}{\alpha} + p_i\right)\right) = \frac{\phi\mu c_t}{k}\frac{\partial\left(-\frac{p_D}{\alpha} + p_i\right)}{\partial t_D t^*} \tag{29}$$

Simplifying (29) gives

$$\frac{1}{r_w^2}\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial p_D}{\partial r_D}\right) = \frac{\phi\mu c_t}{kt^*}\frac{\partial p_D}{\partial t_D}. \tag{30}$$

By setting

$$t^* = \frac{\phi \mu c_t r_w^2}{k}, \tag{31}$$

we obtain

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D \frac{\partial p_D}{\partial r_D}\right) = \frac{\partial p_D}{\partial t_D}. \tag{32}$$

Finally, we determine α from the inner boundary condition (22). Substituting r_D and p_D yields

$$\frac{q\mu}{2\pi kh} = r_D r_w \frac{\partial}{\partial r_D r_w}\left(-\frac{p_D}{\alpha} + p_i\right), \tag{33}$$

which simplifies to

$$\frac{q\mu}{2\pi kh} = -\frac{1}{\alpha}\frac{\partial p_D}{\partial r_D}. \tag{34}$$

By setting

$$\alpha = \frac{2\pi kh}{q\mu} \tag{35}$$

We obtain the dimensionless inner boundary condition

$$\left.\frac{\partial p_D}{\partial r_D}\right|_{r_D=1} = -1. \tag{36}$$

4 Superposition

Solutions to complex problems can be found by adding simple solutions representing the pressure distribution due to wells producing at constant rate at various locations and times. This concept is known as superposition. It is only applicable to linear problems.

Superposition in Time

Assume we have an analytical solution, $p^{const}(q, r, t)$, to the problem of a well producing at a constant rate in a given reservoir. Using superposition in time this solution can be extended to handle a well with a variable flow rate. If a well begins producing at rate q_1 then at time t_1 the rate changes to q_2 the flow rate can be represented as shown in Figure 2.

The analytical solution for the pressure distribution caused by the well producing at variable rate is:

$$p^{var}(r,t) = p^{const}(q_1, r, t) + p^{const}(q_2 - q_1, r, t - t_1)H(t - t_1) \tag{37}$$

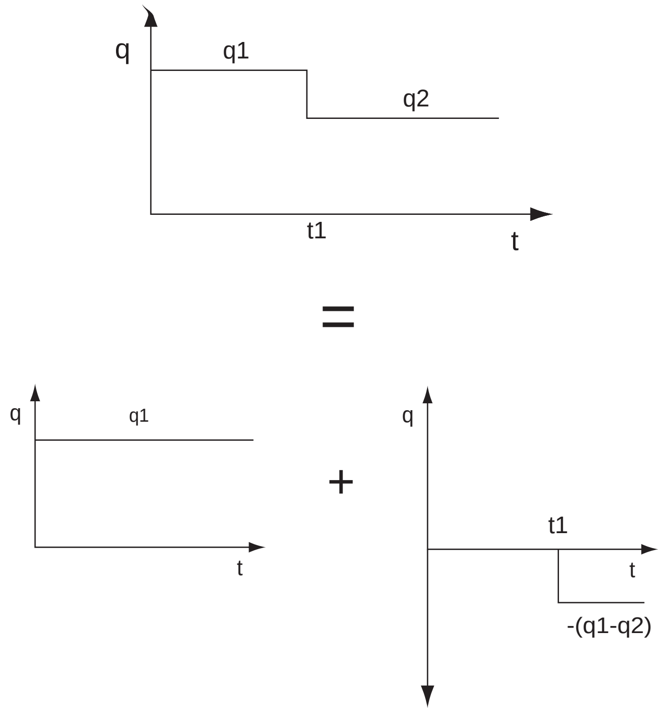

Figure 2: Flow rate variation

where $H(t)$ is the *Heaviside function*, defined by

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} . \tag{38}$$

We will see this function many times throughout this course.

Superposition in Space

Production from multiple wells can be handled using superposition also. Suppose again we have a solution $p^{const}(q,r,t)$ for the pressure distribution due to a well located at the origin, flowing at rate q. The solution for a a reservoir containing two wells as shown in Figure 3 can be generated by summing this solution as follows:

$$p(r,t) = p^{const}(q_1, r_1, t) + p^{const}(q_2, r_2, t) \tag{39}$$

where

$$r_i = \sqrt{(x-x_i)^2 + (y-y_i)^2}, \quad i = 1, 2. \tag{40}$$

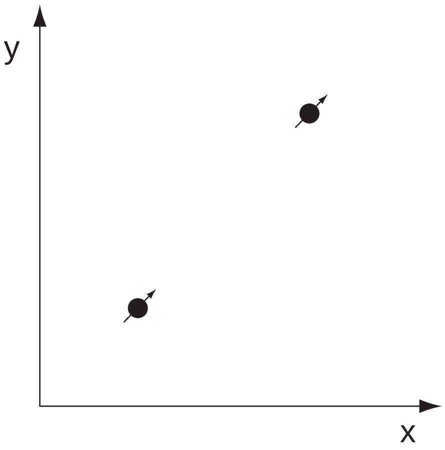

Figure 3: Well configuration

Using Superposition to Handle Boundary Conditions

Superposition in space can be used to impose constant pressure and/or closed boundary conditions. To do so fictitious wells known as image wells are placed in the reservoir in such a way that their effect on the pressure distribution creates the boundary condition. If multiple boundary conditions are involved this can lead to an array of images wells whose contribution to the reservoir pressure distribution is summed. Examples of the use of image wells are shown in Figures 4 and 5.

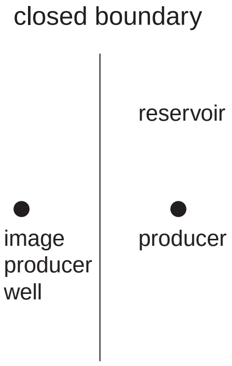

Figure 4: Closed boundary

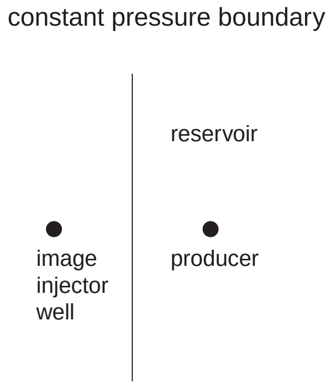

Figure 5: Constant pressure boundary

4.1 Well Boundary Conditions when Superposition is Applied

The superposition theorem guarantees the pressure distribution obtaining by summing simple solutions will satisfy the pressure equation. The boundary condition at the well however requires careful consideration.

4.1.1 Wells Controlled by Bottom Hole Pressure

If a reservoir contains two wells with specified bottom hole pressures p_1 and p_2 a pressure solution can be obtained by summing two solutions for a single well at specified well pressure. This solution will satisfy the pressure equation. However this solution **will not** satisfy the required bottom hole pressures at the wells. If both wells are producers then there will be additional drawdowns at each well due to production in the other well, because the solution for each well will not be nonzero at the location of the other wells. However, if the wells are far apart, this effect is likely to be small.

4.1.2 Wells with a Specified Flow Rate

A solution for a reservoir with multiple wells with specified flow rates can be generated from solutions for a single well. Unlike the case of bottom hole pressure controlled wells this solution **will** satisfy the flow rate boundary condition at each well. This is possible because each superposed solution conserves mass locally so it does not add any extra flow at the well locations.