Jim Lambers
ENERGY 281
Spring Quarter 2007-08
Lecture 2 Notes

1 Separation of Variables

In the previous lecture, we learned how to derive a PDE that describes fluid flow. Now, we will learn a number of analytical techniques for solving such an equation. The first such technique is called *separation of variables*, and it is useful for PDEs on bounded spatial domains with constant coefficients.

Let K be a positive constant. We will solve a diffusion equation in one space dimension:

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{1}$$

with initial condition

$$u(x,0) = f(x), \quad 0 < x < L, \tag{2}$$

and *Dirichlet* boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0. \tag{3}$$

In separation of variables, we assume that the solution $u(x,t)$ has the form

$$u(x,t) = \sum_{j=1}^{\infty} b_j u_j(x,t), \tag{4}$$

where each b_j is a constant, and each function $u_j(x,t)$ can be written as a product of single-variable functions:

$$u_j(x,t) = M_j(x)N_j(t). \tag{5}$$

In order for such a representation of the solution to be valid, each function $u_j(x,t)$ should be a solution of the PDE (1), and satisfy the boundary conditions (3). The constants b_j will be chosen so that the initial condition (2) will be satisfied.

Substituting $u_j(x,t)$ for $u(x,t)$ into (1) and (3), we obtain

$$M_j(x)N_j'(t) = KM_j''(x)N_j'(t) \tag{6}$$

and

$$M_j(0) = M_j(L) = 0. \tag{7}$$

At any point (x, t) such that $u_j(x, t)$ is nonzero, we can divide through (6) by $u_j(x, t)$ and obtain

$$\frac{N_j'(t)}{KN_j(t)} = \frac{M_j''(x)}{M_j(x)}. \tag{8}$$

The left side of this equation is a function of t, while the right side is a function of x. Since they must be equal for all points (x, t) such that $u_j(x, t) \neq 0$, we conclude that both sides must be equal to a constant. That is,

$$\frac{N_j'(t)}{KN_j(t)} = \frac{M_j''(x)}{M_j(x)} = -\lambda, \tag{9}$$

where $-\lambda$ is called the *separation constant*. As we will see, it is merely for convenience that we include the minus sign.

We can now solve for $M_j(x)$ and $N_j(t)$ independently of one another. $M_j(x)$ is a solution of the boundary value problem

$$M_j''(x) + \lambda M_j(x) = 0, \tag{10}$$

with boundary conditions (7). If $\lambda = 0$, then $M_j(x)$ is a linear function, but because of the boundary conditions, it follows that $M_j(x) \equiv 0$ is the only possible solution. On the other hand, if $\lambda < 0$, then (10) is satisfied by $M_j(x) = A\exp[\sqrt{-\lambda}x]$, where A is a constant, but because exponential functions are monotonically increasing or decreasing, it is not possible to find a nonzero function of this form that satisfies the boundary conditions.

The only remaining possibility is $\lambda > 0$. In this case, the solution $M_j(x)$ has the form

$$M_j(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x). \tag{11}$$

Setting $x = 0$, we obtain $M_j(0) = B$, so we must have $B = 0$ in order to satisfy the boundary conditions. Setting $x = L$ yields

$$M_j(L) = A\sin(\sqrt{\lambda}L), \tag{12}$$

so to satisfy the condition $M_j(L) = 0$, we must have $\sqrt{\lambda}L = k\pi$, for some integer k. That is, $\lambda = (k\pi/L)^2$ and $M_j(x) = A\sin(k\pi x/L)$.

If $k = 0$, this yields the trivial solution $M_j(x) \equiv 0$. Because sine is an *odd function*, meaning that $\sin(-x) = -\sin x$ for any x, it follows that we can obtain all of the possible *linearly independent* solutions (meaning, among other things, that no two solutions are scalar multiples of each other) by only considering the case where k is a positive integer. Since we are indexing the functions $M_j(x)$ by the positive integers, it is natural to use $k = j$ and describe the set of all solutions to (10), (7) as follows:

$$M_j(x) = A_j \sin\left(\frac{j\pi x}{L}\right). \tag{13}$$

We can use any nonzero value for the constant A_j, so for convenience, we choose $A_j = 1$ for each j. This yields

$$M_j(x) = \sin\left(\frac{j\pi x}{L}\right), \quad j = 1, 2, \ldots. \tag{14}$$

Next we solve, for each positive integer j,

$$N_j'(t) + \lambda_j K N_j(t) = 0. \tag{15}$$

We obtain

$$N_j(t) = B_j \exp[-\lambda_j K t], \tag{16}$$

where B_j is an arbitrary (nonzero) constant. For convenience, we set $B_j = 1$ for all j.

2 Fourier Sine Series

Now, we have

$$\begin{aligned} u(x,t) &= \sum_{j=1}^{\infty} u_j(x,t) \\ &= \sum_{j=1}^{\infty} b_j M_j(x) N_j(t) \\ &= \sum_{j=1}^{\infty} b_j \sin\left(\frac{j\pi x}{L}\right) \exp\left[-\left(\frac{j\pi}{L}\right)^2 Kt\right]. \end{aligned}$$

We already know that this function satisfies the PDE (1) and the boundary conditions (3), but now we must choose the constants b_j so that the initial condition (2) is also satisfied. Setting $t = 0$ yields

$$f(x) = \sum_{j=1}^{\infty} b_j \sin\left(\frac{j\pi x}{L}\right). \tag{17}$$

To find each constant b_k, we multiply both sides of (17) by $M_k(x)$ and integrate from 0 to L. That is, we take the *inner product* of the functions on both sides of the equation with $M_k(x)$:

$$(M_k(x), f(x)) = \sum_{j=1}^{\infty} b_j (M_k(x), M_j(x)), \tag{18}$$

where the inner product $(u(x), v(x))$ of two real-valued functions $u(x)$ and $v(x)$ defined on the interval $0 < x < L$ is defined by

$$(u(x), v(x)) = \int_0^L u(x)v(x)\,dx. \tag{19}$$

The functions $M_j(x)$ are *orthogonal* to one another. That is,

$$(M_k(x), M_j(x)) = 0, \quad k \neq j. \tag{20}$$

To see this, note that if k and j are positive integers such that $k \neq j$, we can use a *product-to-sum identity* to obtain

$$\begin{aligned}
(M_k(x), M_j(x)) &= \int_0^L \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{j\pi x}{L}\right) dx \\
&= \int_0^{\pi} \frac{L}{\pi} \sin ku \sin ju \, du \\
&= \frac{L}{2\pi} \int_0^{\pi} \cos[(k-j)u] - \cos[(k+j)u]\, du \\
&= \frac{L}{2\pi} \left[\frac{\sin[(k-j)u]}{k-j} - \frac{\sin[(k+j)u]}{k+j} \Bigg|_0^{\pi} \right] \\
&= \frac{L}{2\pi} \frac{(k+j)\sin[(k-j)]\pi] - (k-j)\sin[(k+j)\pi]}{k^2 - j^2} \\
&= 0,
\end{aligned}$$

since sine is zero at any multiple of π. On the other hand, if $k = j$, then we have $(M_k(x), M_k(x)) = L/2$.

Therefore, all of the terms on the right side of (18) vanish except for the Kth term, which yields

$$b_k = \frac{(M_k(x), f(x))}{(M_k(x), M_k(x))}, \quad k = 1, 2, \ldots. \tag{21}$$

The constants b_k, for $k = 1, 2, \ldots$, define the *Fourier sine series* of $f(x)$:

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{L}\right), \tag{22}$$

where

$$b_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right), \quad k = 1, 2, \ldots. \tag{23}$$

Computing the coefficients of this series completes the solution process.

3 Neumann Boundary Conditions

Now, suppose that instead of the Dirichlet boundary conditions (3), we have *Neumann* boundary conditions

$$u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0, \tag{24}$$

where u_x is a common shorthand for $\frac{\partial u}{\partial x}$. Then, we can proceed as before, but then $M_j(x)$ satisfies the boundary conditions

$$M_j'(0) = 0, \quad M_j'(L) = 0. \tag{25}$$

The solution to (10) has the form (11), but this time, $A = 0$ and B is nonzero. We still have $\lambda = (k\pi/L)^2$, but this time, $k = 0$ yields $M_j(x) = B$, so we have the solutions

$$M_j(x) = \cos\left(\frac{j\pi x}{L}\right), \quad j = 0, 1, 2, \ldots. \tag{26}$$

The solution to (1), (2), (24) has the form

$$u(x,t) = \sum_{j=0}^{\infty} a_j M_j(x) N_j(t) \tag{27}$$

where $M_j(x)$ is defined in (26) and $N_j(t)$ is defined in (16). The constants a_j are given by the *Fourier cosine series* of $f(x)$,

$$f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos\left(\frac{j\pi x}{L}\right), \tag{28}$$

where

$$a_j = \frac{(M_j(x), f(x))}{(M_j(x), M_j(x))} = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{j\pi x}{L}\right)\,dx, \quad j = 0, 1, 2, \ldots. \tag{29}$$

There are also *mixed* boundary conditions, which are linear combinations of Dirichlet and Neumann conditions, which can be addressed in a similar manner.

4 Generalizations

Separation of variables can be applied to other PDEs on bounded spatial domains besides diffusion problems of the form (1). For example, consider the *wave equation*, also known as the *telegraph equation*:

$$\frac{\partial^2 u}{\partial t^2} = K\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{30}$$

where K is a positive constant, with initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L, \tag{31}$$

with either Dirichlet or Neumann boundary conditions.

In this case, the ODE (10) that characterizes $M_j(x)$ is the same, but to obtain $N_j(t)$, we must solve

$$N_j''(t) + \lambda N(t) = 0, \tag{32}$$

and since the initial conditions are not yet taken into account, $N_j(t)$ is a general linear combination:

$$N_j(t) = a_j \cos(\sqrt{\lambda}t) + b_j \sin(\sqrt{\lambda}t), \tag{33}$$

where the two sets of constants $\{a_j\}$ and $\{b_j\}$ must be chosen in order to satisfy the initial conditions (31). As before, the Fourier sine or cosine series of $f(x)$ and $g(x)$ need to be computed, depending on the boundary conditions.

We can also use separation of variables to solve *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < L_x, \quad 0 < y < L_y, \tag{34}$$

with boundary conditions that are homogeneous in one of the two dimensions and inhomogeneous in the other. For example, Dirichlet boundary conditions could be

$$u(x,0) = 0, \quad u(x,L_y) = 0, \quad u(0,y) = f(y), \quad u(L_x,y) = g(y) \tag{35}$$

or

$$u(x,0) = f(x), \quad u(x,L_y) = g(x), \quad u(0,y) = 0, \quad u(L_x,y) = 0. \tag{36}$$

If the boundary conditions at $x = 0$ and $x = L_x$ are homogeneous (for example, if the conditions (36) apply), then they must be imposed on each $M_j(x)$. Then, the ODEs to be solved are

$$M_j''(x) + \lambda M_j(x) = 0, \quad N_j''(y) - \lambda N_j(y) = 0, \tag{37}$$

where $\lambda \geq 0$. The arbitrary constants in $N_j(y)$ are obtained from the Fourier series of the functions from the boundary conditions at $y = 0$ and $y = L_y$. Because there are two arbitrary constants for each function $N_j(y)$, and two boundary conditions that they must satisfy, it is necessary to solve a 2×2 system of linear equations for each j. The opposite procedure is followed if the boundary conditions at $y = 0$ and $y = L_y$ are homogeneous instead: the homogeneous boundary conditions are imposed on each $N_j(y)$, and Fourier series are used to determine the arbitrary constants in each $M_j(x)$. In this case, the ODEs to be solved are

$$M_j''(x) - \lambda M_j(x) = 0, \quad N_j''(y) + \lambda N_j(y) = 0, \tag{38}$$

where $\lambda \geq 0$.

Separation of variables can be used, indirectly, to solve time-dependent inhomogeneous PDE such as

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + F(x,t), \quad 0 < x < L, \quad t > 0, \tag{39}$$

with initial conditions (2) and (3). The function $F(x,t)$ is called a *source term* or *forcing term*. By *Duhamel's principle*, the solution to this problem is

$$u(x,t) = u_h(x,t) + \int_0^t v_\tau(x; t-\tau)\, d\tau, \tag{40}$$

where $u_h(x,t)$ is the solution to the corresponding homogeneous problem (1), (2), (3), and $v_\tau(x,t)$ is a *one-parameter family* of solutions of the set of homogeneous problems (1), (3) with initial conditions

$$v_\tau(x,0) = F(x,\tau), \quad 0 < x < L. \tag{41}$$

All of the homogeneous problems needed to solve the original inhomogeneous problem can be solved using separation of variables.

It should be noted that all of the problems we have considered in this lecture have homogeneous boundary conditions; that is, either the values of the solution or certain derivatives of the solution must be equal to zero on the boundary. If a PDE has inhomogeneous boundary conditions, separation of variables cannot be used directly. However, if one can find a function that satisfies the inhomogeneous boundary conditions and subtract it from the (unknown) solution, then the difference satisfies the same PDE, albeit with a source term, and homogeneous boundary conditions. Then, separation of variables can be applied, in conjunction with Duhamel's principle if the problem is time-dependent.

5 Classification of Second-Order PDEs

Consider a general second-order PDE of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + F = 0. \tag{42}$$

In many cases, the variable y is actually used to denote time, in which case t is used instead, but we use y for uniformity.

The equation is classified using the sign of the coefficients A, B and C:

hyperbolic	if $B^2 - 4AC > 0$
parabolic	if $B^2 - 4AC = 0$
elliptic	if $B^2 - 4AC < 0$

These names arise from the classification of conic sections from general quadratic equations of two variables. Based on this classification scheme, the equation (1) is parabolic.

From this classification scheme, we see that the diffusion equation is parabolic, the wave equation is hyperbolic, and Laplace's equation is elliptic. Note that we are not assuming that the coefficients A, B and C are constant, so if they do depend on x and y, it is possible that an equation may have different classifications at different points in its domain.