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ENERGY 281
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Lecture 3 Notes

These notes are based on Rosalind Archer's PE281 lecture notes, with some revisions by Jim Lambers.

1 Introduction

The Fourier transform is an integral transform. When viewed in the context of signal processing the application of the Fourier transform takes a function from real-space to frequency-space (see later examples). The Fourier transform is defined by:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixs}\, dx \tag{1}$$

The inverse transform is defined in a similar manner:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{ixs}\, ds \tag{2}$$

We will also use the notation $\hat{f}(s)$ for $F(s)$. The Fourier transform exists if $f(x)$ and $f'(x)$ are at least piecewise continuous and the following integral exists:

$$\int_{-\infty}^{\infty} |f(x)|\, dx \tag{3}$$

There are also some alternative definitions:

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-ixs}\, dx \tag{4}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{ixs}\, ds \tag{5}$$

and

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi xs}\, dx \tag{6}$$

$$f(x) = \int_{-\infty}^{\infty} F(s) e^{i2\pi xs}\, ds \tag{7}$$

Example Let

$$f(x) = e^{-x^2}. \tag{8}$$

Then

$$\begin{aligned}
F(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{-isx}\, dx & (9)\\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2+ixs)}\, dx & (10)\\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2+ixs-s^2/4+s^2/4)}\, dx & (11)\\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+is/2)^2-s^2/4}\, dx & (12)\\
&= \frac{1}{\sqrt{2\pi}} e^{-s^2/4} \int_{-\infty}^{\infty} e^{-(x+is/2)^2}\, dx & (13)\\
&= \frac{1}{\sqrt{2\pi}} e^{-s^2/4} \int_{-\infty+is/2}^{\infty+is/2} e^{-\xi^2}\, d\xi & (14)\\
&= \frac{1}{\sqrt{2\pi}} e^{-s^2/4} \int_{-\infty}^{\infty} e^{-x^2}\, dx & (15)\\
&= \frac{1}{\sqrt{2\pi}} e^{-s^2/4} \sqrt{\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)} & (16)\\
&= \frac{1}{\sqrt{2\pi}} e^{-s^2/4} \sqrt{\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right)} & (17)\\
&= \frac{1}{\sqrt{2\pi}} e^{-s^2/4} \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy} & (18)\\
&= \frac{1}{\sqrt{2\pi}} e^{-s^2/4} \sqrt{\int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r\, dr\, d\theta} & (19)\\
&= \frac{1}{\sqrt{2\pi}} e^{-s^2/4} \sqrt{\frac{1}{2}\int_{0}^{2\pi}\int_{0}^{\infty} e^{-u}\, du\, d\theta} & (20)\\
&= \frac{1}{\sqrt{2\pi}} e^{-s^2/4} \sqrt{\pi \int_{0}^{\infty} e^{-u}\, du} & (21)\\
&= \frac{1}{\sqrt{2\pi}} e^{-s^2/4} \sqrt{-\pi e^{-u}\big|_0^{\infty}} & (22)\\
&= \frac{1}{\sqrt{2\pi}} e^{-s^2/4} \sqrt{\pi} & (23)\\
&= \frac{1}{\sqrt{2}} e^{-s^2/4}, & (24)
\end{aligned}$$

where, in equation (14), $\xi = x + is/2$. In (15), we applied the Cauchy integral theorem, applied to a rectangle in the complex plane with vertices $(x, 0)$, $(-x, 0)$, $(x, is/2)$, and $(-x, is/2)$, as $x \to \infty$. In (19), we converted to polar coordinates to obtain an integral that could be evaluated.

The Fourier transform relates a function in real space (either time or distance) to a function in frequency space. This can be seen by recalling:

$$e^{ixs} = \cos(xs) + i\sin(xs) \tag{25}$$

Now consider the inverse transform:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{ixs} ds \tag{26}$$

This integral shows that the Fourier transform breaks a function $f(x)$ into a sum of sines and cosines with frequency s. (Recall the frequency of $f(kx)$ is $|k|$). The amplitude associated with any given frequency is given by $F(s)$.

Example Consider $f = \cos x$. The Fourier transform of f is

$$F(s) = \frac{\sqrt{2\pi}}{2}\delta(-1+s) + \frac{\sqrt{2\pi}}{2}\delta(1+s), \tag{27}$$

where $\delta(s)$ is the *Dirac delta function*, which has the property

$$\int_{-\infty}^{\infty} f(s)\delta(s-c)\, ds = f(c). \tag{28}$$

To verify that $F(s)$ really is the Fourier transform of $f(x)$, we compute its inverse transform:

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} F(s)\, ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \left[\frac{\sqrt{2\pi}}{2}\delta(-1+s) + \frac{\sqrt{2\pi}}{2}\delta(1+s)\right] ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \frac{\sqrt{2\pi}}{2}\delta(-1+s)\, ds + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \frac{\sqrt{2\pi}}{2}\delta(1+s)\, ds \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{isx}\delta(-1+s)\, ds + \frac{1}{2}\int_{-\infty}^{\infty} e^{isx}\delta(1+s)\, ds \\
&= \frac{1}{2}(e^{ix} + e^{-ix}) \\
&= \frac{1}{2}(\cos x + i\sin x + \cos(-x) + i\sin(-x)) \\
&= \frac{1}{2}(\cos x + \cos x) \\
&= \cos x.
\end{aligned} \tag{29}$$

The original function $f(x)$ and its Fourier transform $F(s)$ are shown in Figures 1 and 2, respectively.

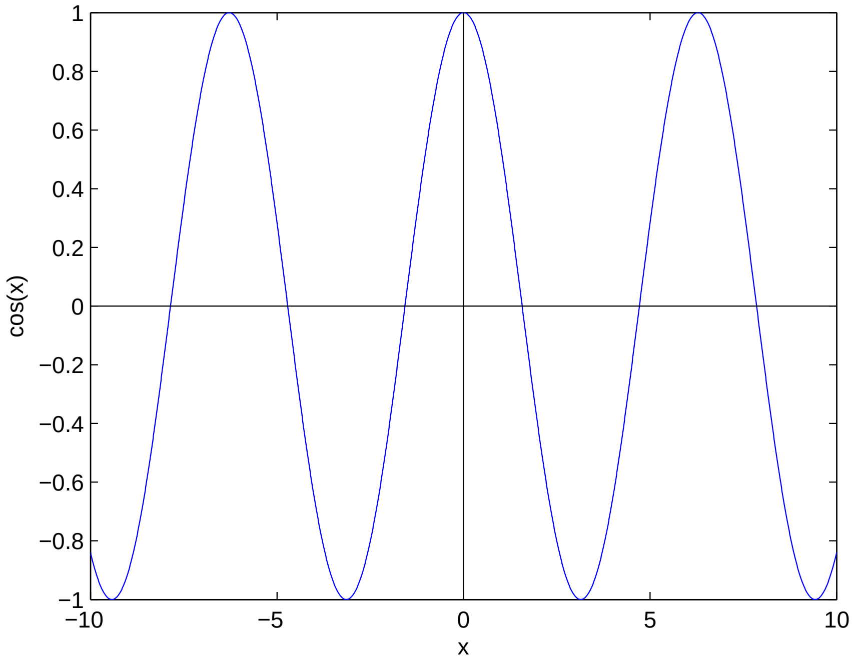

Figure 1: $f(x) = \cos x$

2 Fourier Transform Theorems

In stating and using the following theorems, which greatly simplify the process of computing transforms and inverse transforms of functions, we will use the following notation: given a function $f(x)$, $\mathcal{F}(f(x))$ refers to its Fourier transform, commonly denoted by $F(s)$. That is, $\mathcal{F}$ is a function whose input is a function of x, and whose output is its transform, a function of s.

2.1 Theorem 1 - Linearity

$$\mathcal{F}(f(x) + g(x)) = \mathcal{F}(f(x)) + \mathcal{F}(g(x)) \tag{30}$$

and

$$\mathcal{F}(cf(x)) = c\mathcal{F}(f(x)) \tag{31}$$

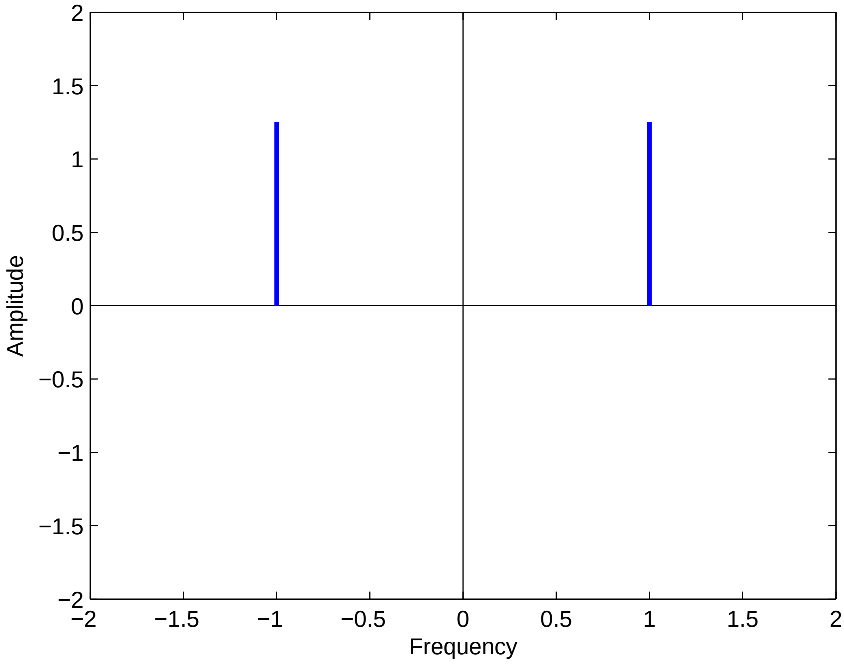

Figure 2: Frequency spectrum of $f(x) = \cos x$

2.2 Theorem 2 - Shift Theorem

If

$$\mathcal{F}(f(x)) = F(s) \tag{32}$$

then

$$\mathcal{F}(f(x-a)) = e^{-isa}F(s). \tag{33}$$

Proof:

$$\begin{aligned}
\mathcal{F}(f(x-a)) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)e^{-isx}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)e^{-is(x-a)-isa}\,dx \\
&= e^{-isa}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)e^{-is(x-a)}\,dx \\
&= e^{-isa}F(s)
\end{aligned} \tag{34}$$

2.3 Theorem 3 - Similarity Theorem

If

$$\mathcal{F}(f(x)) = F(s) \tag{35}$$

then

$$\mathcal{F}(f(ax)) = \frac{1}{|a|}F\left(\frac{s}{a}\right) \tag{36}$$

2.4 Theorem 4 - Convolution Theorem

If

$$\mathcal{F}(f(x)) = F(s) \tag{37}$$

and

$$\mathcal{F}(g(x)) = G(s) \tag{38}$$

then

$$\mathcal{F}(f(x) \star g(x)) = \sqrt{2\pi}F(s)G(s) \tag{39}$$

where

$$(f(x) \star g(x))(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt \tag{40}$$

is the *convolution* of f and g.

2.5 Theorem 5 - Parseval's theorem

If

$$\mathcal{F}(f(x)) = F(s) \tag{41}$$

then

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds. \tag{42}$$

2.6 Theorem 6 - Derivatives

If

$$\mathcal{F}(f(x)) = F(s) \tag{43}$$

then

$$\mathcal{F}(f^{(n)}(x)) = (is)^n F(s) \tag{44}$$

where

$$f^{(n)}(x) = \frac{d^n}{dx^n}f(x). \tag{45}$$

Note that this assumes the values of the derivatives vanish at $\pm\infty$.

3 Fourier Sine and Cosine Transforms

The Fourier transform is defined as:

$$\begin{aligned}\mathcal{F}(f(x)) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)[\cos(-sx)+i\sin(-sx)]\,dx. \end{aligned} \tag{46}$$

Now consider a case where $f(x)$ is the sum of an even and an odd function, $f_e(x)$ and $f_o(x)$. Recall that if $f(x)$ is an odd function,

$$f(-x) = -f(x). \tag{47}$$

$\sin x$ is an example of an odd function. On the other hand, if $f(x)$ is an even function,

$$f(-x) = f(x). \tag{48}$$

$\cos x$ is an example of an even function. Any function $f(x)$ can be written as the sum of an even and odd function. Specifically,

$$f(x) = f_e(x) + f_o(x), \tag{49}$$

where

$$f_e(x) = \frac{f(x)+f(-x)}{2}, \quad f_o(x) = \frac{f(x)-f(-x)}{2}. \tag{50}$$

With $f(x)$ defined as the sum of an even and odd function, the Fourier transform of $f(x)$ becomes

$$\begin{aligned}F(f(x)) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f_e(x)+f_o(x))\cos sx\,dx - \\ & i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f_e(x)+f_o(x))\sin sx\,dx. \end{aligned} \tag{51}$$

Now take into account the way products of even and odd functions behave:

- The product of two even functions, or two odd functions, is an even function.
- The product of an even function and an odd function is an odd function.

Also note the following integral:

$$\int_{-\infty}^{\infty} f_o(x)dx = 0 \tag{52}$$

Now substitute these relationships into (51):

$$\begin{aligned} \mathcal{F}(f(x)) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_e(x) \cos sx \, dx - i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_o(x) \sin sx \, dx & (53) \\ &= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f_e(x) \cos sx \, dx - \frac{2i}{\sqrt{2\pi}} \int_{0}^{\infty} f_o(x) \sin sx \, dx. & (54) \end{aligned}$$

The fact that the Fourier transform splits into two terms (sine and cosine) motivates the definition of the sine and cosine transforms $\mathcal{F}_s$ and $\mathcal{F}_c$:

$$\mathcal{F}_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos xs \, dx = F_c(s) \tag{55}$$

$$\mathcal{F}_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin xs \, dx = F_s(s) \tag{56}$$

The inverse transforms are defined as follows:

$$\mathcal{F}_c^{-1}(F_c(s)) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c(s) \cos xs \, ds = f(x) \tag{57}$$

$$\mathcal{F}_s^{-1}(F_s(s)) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_s(s) \sin xs \, ds = f(x) \tag{58}$$

The following rules for transforming second derivatives are especially useful:

$$\mathcal{F}_c(f'') = -\sqrt{\frac{2}{\pi}} f'(0) - s^2 \mathcal{F}_c(f), \tag{59}$$

$$\mathcal{F}_s(f'') = s\sqrt{\frac{2}{\pi}} f(0) - s^2 \mathcal{F}_s(f). \tag{60}$$

The use of sine and cosine transforms simplifies the transform procedure when transforming even and odd functions, because the Fourier sine transform of an even function is zero, as is the Fourier cosine transform of an odd function. It follows that the sine and cosine transforms can be used in place of the full Fourier transform for problems with

- semi-infinite domains,
- differential equations that have *only* even orders of derivatives,
- either f or f' specified at the boundary.

4 Example 1: 1D Pressure Diffusion

Consider a one-dimensional problem governed by the pressure equation in dimensionless form,

$$\frac{\partial^2 p_D}{\partial x_D^2} = \frac{\partial p_D}{\partial t_D} \tag{61}$$

The boundary conditions are

$$p_D(0, t_D) = 1, \quad \lim_{x_D \to \infty} p_D(x_D, t_D) = 0, \tag{62}$$

and the initial condition is

$$p_D(x_D, 0) = 0. \tag{63}$$

Since the pressure and not the pressure derivative is set on the boundary, we should use the sine transform. The choice of transform is made according to equations (59) and (60), which relate the transform of the second derivative to the boundary conditions.

First, we apply the Fourier sine transform to the spatial variable in order to convert the partial differential equation to an ordinary differential equation:

$$s\sqrt{\frac{2}{\pi}} p_D(x_D = 0, t_D) - s^2 \hat{p}_D = \frac{\partial \hat{p}_D}{\partial t_D} \tag{64}$$

$$\implies \quad \frac{\partial \hat{p}_D}{\partial t_D} + s^2 \hat{p}_D = \sqrt{\frac{2}{\pi}} s. \tag{65}$$

This equation can be solved using an *integrating factor*. Recall that for a general linear first-order ODE of the form

$$\frac{dy}{dt} + P(t)y = Q(t) \tag{66}$$

we can multiply through by the integrating factor $\exp\left[\int^t P(s)\, ds\right]$ to obtain

$$e^{\int^t P(s)\, ds} \frac{du}{dt} + P(t) e^{\int^t P(s)\, ds} y = e^{\int^t P(s)\, ds} Q(t) \tag{67}$$

$$\implies \quad \frac{d}{dt}\left(e^{\int^t P(s)\, ds} y \right) = e^{\int^t P(s)\, ds} Q(t) \tag{68}$$

$$\implies \quad y(t) e^{\int^t P(s)\, ds} = \int^t Q(\tau) e^{\int^\tau P(s)\, ds} d\tau + C. \tag{69}$$

Applying this technique to the transformed equation (65) with $y(t) = \hat{p}_D(s,t_D)$, $P(t) = s^2$ and $Q(t) = \sqrt{2/\pi}s$ yields

$$\hat{p}_D(s,t_D)e^{s^2 t_D} = \int_0^{t_D} \sqrt{\frac{2}{\pi}} s e^{s^2 \tau} d\tau \tag{70}$$

$$\begin{aligned} \Longrightarrow \quad \hat{p}_D(s,t_D) &= \int_0^{t_D} \sqrt{\frac{2}{\pi}} s e^{-s^2(t_D-\tau)}\, d\tau \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{s}(1 - e^{-s^2 t_D}). \end{aligned} \tag{71}$$

Now invert the Fourier sine transform to find p_D:

$$\begin{aligned} p_D(x_D,t_D) &= \mathcal{F}_s^{-1}(\hat{p}_D(s,t_D)) \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{p}_D(s,t_D) \sin(sx_D)\, ds \\ &= \frac{2}{\pi} \int_0^\infty \frac{1}{s}(1 - e^{-s^2 t_D}) \sin(sx_D)\, ds \\ &= 1 - Erf\left(\frac{x_D}{\sqrt{4t_D}}\right) = Erfc\left(\frac{x_D}{\sqrt{4t_D}}\right) \end{aligned} \tag{72}$$

where $Erf(z)$ and $Erfc(z)$ are the error function and the complimentary error function, respectively, defined by

$$Erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt \tag{73}$$

$$Erfc(z) = 1 - Erf(z) \tag{74}$$