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ENERGY 281
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Lecture 4 Notes

These notes are based on Rosalind Archer's PE281 lecture notes, with some revisions by Jim Lambers.

1 Example 2: Heat Equation

Consider the diffusion of heat in a one-dimensional bar. We'll consider an infinitely long bar so the full Fourier transform is required. The governing equation is:

$$\kappa^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \tag{1}$$

The boundary conditions are:

$$T(x \pm \infty, t) = T'(x \pm \infty, t) = 0 \tag{2}$$

The initial condition is a prescribed temperature that varies in space:

$$T(x, t = 0) = T_0(x) \tag{3}$$

First consider the Fourier transform of the spatial derivatives:

$$\begin{aligned} \mathcal{F}(f'(x)) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} f'(x)\, dx \\ &= \frac{1}{\sqrt{2\pi}} f(x) e^{-isx}|_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-is) f(x) e^{-isx}\, dx \\ &= (is)\mathcal{F}(f(x)) \end{aligned} \tag{4}$$

since $f(x)$ vanishes at $\pm\infty$. Similar arguments require $f'(x)$ vanishes at $\pm\infty$.

The transformed differential equation is

$$\frac{\partial \hat{T}}{\partial t} + \kappa^2 s^2 \hat{T} = 0 \tag{5}$$

which has the solution

$$\hat{T}(s, t) = c_1(s) e^{-(\kappa s)^2 t} \tag{6}$$

for some function $c_1(s)$. Now consider the initial condition

$$\hat{T}(s, t=0) = \hat{T}_0(s). \tag{7}$$

It follows from (6) that $c_1(s) = \hat{T}_0(s)$ and therefore

$$\hat{T}(s,t) = \hat{T}_0(s) e^{-(\kappa s)^2 t}. \tag{8}$$

Now invert to find T:

$$T(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \hat{T}_0(s) e^{-(\kappa s)^2 t}\, ds \tag{9}$$

The transform of T_0 is:

$$\hat{T}_0(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} T_0(\lambda) e^{-is\lambda}\, d\lambda. \tag{10}$$

Substituting this transform into (9) yields

$$\begin{aligned} T(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} \int_{-\infty}^{\infty} e^{-is\lambda} T_0(\lambda) e^{-(\kappa s)^2 t}\, d\lambda\, ds & (11)\\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} T_0(\lambda) \int_{-\infty}^{\infty} e^{-is(\lambda - x) - (\kappa s)^2 t}\, ds\, d\lambda & (12)\\ &= \frac{1}{\sqrt{4\kappa^2 \pi t}} \int_{-\infty}^{\infty} T_0(\lambda) e^{-\frac{(\lambda - x)^2}{4\kappa^2 t}}\, d\lambda. & (13) \end{aligned}$$

2 Example 3: Elliptic Problem

Consider a steady-state problem in a two-dimensional semi-infinite domain governed by:

$$\nabla^2 p = 0 \tag{14}$$

The boundary conditions are:

$$p(0, y) = 0 \tag{15}$$

$$\lim_{x \to \infty} p(x, y) = 0 \tag{16}$$

$$p(x, 0) = f(x) \tag{17}$$

$$p(x, a) = 0 \tag{18}$$

Since the domain is semi-infinite and the pressure is specified on the boundary, we will use the sine transform to transform the differential equation:

$$\mathcal{F}_s\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) = 0 \tag{19}$$

$$\Rightarrow s\sqrt{\frac{2}{\pi}}p(0, y) - s^2\hat{p} + \frac{\partial^2 \hat{p}}{\partial y^2} = 0 \tag{20}$$

$$\Rightarrow \frac{\partial^2 \hat{p}}{\partial y^2} - s^2\hat{p} = 0 \tag{21}$$

This equation can be solved for $\hat{p}$ to give:

$$\hat{p}(s, y) = c_1(s)cos(isy) + c_2(s)sin(isy) \tag{22}$$

Now use the boundary conditions to determine $c_1(s)$ and $c_2(s)$:

$$\mathcal{F}_s(p(x, y = 0)) = \mathcal{F}_s(f(x)) = F_1(s) \tag{23}$$

$$\mathcal{F}_s(p(x, y = a)) = 0 \tag{24}$$

After some algebra we can show:

$$\hat{p}(s, y) = F_1(s)\frac{\sinh(s(a - y))}{\sinh(sa)}, \tag{25}$$

where

$$\sinh(z) = \frac{e^z - e^{-z}}{2} = -i\sin(iz). \tag{26}$$

Now invert to find p:

$$p(x, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_1(s)\frac{\sinh(s(a - y))}{\sinh(sa)}\sin(xs)\, ds \tag{27}$$

where

$$F_1(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(\lambda)\sin(s\lambda)\, d\lambda \tag{28}$$

is the Fourier sine transform of $f(x)$. Substituting $F_1(s)$ into the expression for $p(x, y)$ gives:

$$p(x, y) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(\lambda)\frac{\sinh(s(a - y))}{\sinh(sa)}\sin(s\lambda)\sin(sx)\, d\lambda\, ds, \tag{29}$$

which, due to its complexity, will have to be inverted numerically. We will discuss this later.

3 Radial Problems

All of the examples presented so far have been for infinite domains, or semi-infinite domains with linear boundaries. However, radial problems are often of more interest to petroleum engineers. Is the Fourier transform helpful in these cases? Consider the problem

$$\frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial p_D}{\partial r_D} = \frac{\partial p_D}{\partial t_D} \tag{30}$$

The sine and cosine transforms won't work, because the pressure equation in radial coordinates includes both even and odd orders of derivatives. The full Fourier transform is a candidate, if we consider applying it to the spatial variable r_D. When transforming the spatial derivatives, we will require the behavior of the pressure at $\pm\infty$. Ideally, the pressure and its first derivative should vanish at $\pm\infty$. This may be the case at $r = +\infty$, but it is much harder to make that claim at $r = -\infty$. Applying the full Fourier transform in time is another option. However, transforming the time derivative requires the behavior of the pressure at $t = -\infty$. This is not such a problem, since it is likely $p = p_i$ would be suitable. However, now the boundary conditions become time dependent if the flow begins at $t = 0$.

The *Hankel transform*

$$H_\nu(\lambda) = \int_0^\infty r J_\nu(\lambda r) f(r)\, dr \tag{31}$$

$$f(r) = \int_0^\infty \lambda J_\nu(\lambda r) H_\nu(\lambda)\, d\lambda, \tag{32}$$

where J_ν is a *Bessel function of the first kind*, is better suited to radial problems. We'll see Bessel functions later in this course.

4 Inverting Fourier Transforms Numerically

Since the limits of the inversion integral are real, standard numerical integration routines can be used to evaluate the integral that defines the inverse Fourier transform. Due to the oscillatory nature of the integrand, however, many function evaluations may be required, especially for large values of s. If the Fourier transform is being applied to discrete data instead of a function, there are formal algorithms that can be used to perform the inversion. We will discuss this further when we learn about the *discrete Fourier transform.*