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**Lecture 5 Notes**

These notes are based on Rosalind Archer's PE281 lecture notes, with some revisions by Jim Lambers.

# 1 Fourier Series

Recall that in Lecture 2, when we considered PDE on bounded spatial domains, we expressed solutions in terms of a Fourier sine series, in the case of Dirichlet boundary conditions, or Fourier cosine series, in the case of Neumann boundary conditions. Now, suppose that a solution $u(x,t)$ of a PDE on the interval $[0,L]$ is intended to satisfy *periodic boundary conditions*

$$u(0,t) = u(L,t), \quad t > 0. \tag{1}$$

Then, since $\sin x$ and $\cos x$ are both $2\pi$-periodic, the solution can be expressed in terms of both sines and cosines, as follows:

$$u(x,t) = a_0(t) + \sum_{j=1}^{\infty} a_j(t)\cos\frac{2\pi jx}{L} + b_j(t)\sin\frac{2\pi jx}{L}, \tag{2}$$

where, for $j = 1, 2, \ldots$, the coefficients $a_j(t)$ and $b_j(t)$ are defined by

$$a_j(t) = \frac{2}{L}\int_0^L u(x,t)\cos\frac{2\pi jx}{L}\,dx, \quad bj(t) = \frac{2}{L}\int_0^L u(x,t)\sin\frac{2\pi jx}{L}\,dx, \tag{3}$$

and

$$a_0(t) = \frac{1}{L}\int_0^L u(x,t)\,dx. \tag{4}$$

This series representation of $u(x,t)$ is called the *Fourier series* of $u(x,t)$.

## 1.1 The Complex Exponential Form

Alternatively, we can use the relation

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{5}$$

to express the solution in terms of complex exponentials,

$$u(x,t) = \frac{1}{\sqrt{L}}\sum_{\omega=-\infty}^{\infty} \hat{u}(\omega,t)e^{2\pi i\omega x/L}, \tag{6}$$

where

$$\hat{u}(\omega, t) = \frac{1}{\sqrt{L}} \int_0^L e^{-2\pi i \omega x/L} u(x, t)\, dx. \tag{7}$$

Clearly, this series resembles the formula for the inverse Fourier transform, while the formula for the coefficients resembles that of the Fourier transform.

## 1.2 Orthogonality

Like the sines and cosines in (2), the functions $e^{2\pi i \omega x/L}$ are orthogonal with respect to the inner product

$$(u, v) = \int_0^L \overline{u(x)} v(x)\, dx, \tag{8}$$

where the complex conjugation of u(x) is necessary to ensure that the norm $\|\cdot\|$ defined by

$$\|u\| = \sqrt{(u, u)} \tag{9}$$

satisfies one of the essential properties of norms, that the norm of a function must be nonnegative.

Specifically, we have

$$\left(e^{2\pi i \omega x/L}, e^{2\pi i \eta x/L}\right) = \begin{cases} L & \omega = \eta \\ 0 & \omega \neq \eta \end{cases}. \tag{10}$$

This explains the presence of the scaling constant $1/\sqrt{L}$ in (6). It normalizes the functions $e^{2\pi i \omega x/L}$ so that they form an orthonormal set, meaning that they are orthogonal to one another, and have unit norm.

One consequence of this orthonormality is an analogue of Parsevals identity for Fourier series. We say that $f(x)$ is *square-integrable* on $[0, L]$ if

$$\int_0^L |f(x)|^2\, dx < \infty. \tag{11}$$

That is, the above integral must be finite. If such a function is also piecewise continuous, the following identity is satisfied:

$$\sum_{\omega=-\infty}^{\infty} |\hat{f}(\omega)|^2 = \|f\|^2, \tag{12}$$

where the norm $\|\cdot\|$ is as defined in (9).

### 1.3 Convergence

The Fourier series for an $L$-periodic function $f(x)$ will converge to $f(x)$ at any point in $[0, L]$ at which $f$ is continuously differentiable. If $f$ has a jump discontinuity at a point $c$, then the series will converge to $\frac{1}{2}[f(c^+) + f(c^-)]$, where

$$f(c^+) = \lim_{x \to c^+} f(x), \quad f(c^-) = \lim_{x \to c^-} f(x). \tag{13}$$

Convergence of the Fourier series of $f$ is more rapid when $f$ is smooth. In particular, if $f$ is $p$-times differentiable and its $p$th derivative is at least *piecewise* continuous (that is, continuous except possibly for jump discontinuities), then the coefficients of the complex exponential form of the Fourier series satisfy

$$|\hat{f}(\omega)| \leq \frac{C}{|\omega|^{p+1} + 1} \tag{14}$$

for some constant $C$ that is independent of $\omega$.

## 2 The Discrete Fourier Transform

Now, suppose that we define a grid on the interval $[0, L]$, consisting of the $N$ points

$$x_j = j\Delta x, \quad \Delta x = \frac{L}{N}, \quad j = 0, \ldots, N-1. \tag{15}$$

Given an $L$-periodic function $f(x)$, we would like to compute an approximation to its Fourier series of the form

$$f_N(x) = \frac{1}{\sqrt{L}} \sum_{\omega=-N/2+1}^{N/2} e^{2\pi i \omega x/L} \tilde{f}(\omega), \tag{16}$$

where each $\tilde{f}(\omega)$ approximates the corresponding coefficient $\hat{f}(\omega)$ of the true Fourier series. Ideally, this approximate series should satisfy

$$f_N(x_j) = f(x_j), \quad j = 0, 1, \ldots, N-1. \tag{17}$$

That is, $f_N(x)$ should be an *interpolant* of $f(x)$, with the $N$ points $x_j$, $j = 0, 1, \ldots, N-1$, as the interpolation points.

The problem of finding this interpolant, called the *Fourier interpolant* of $f$, has a unique solution that can easily be computed. The coefficients $\tilde{f}(\omega)$

are obtained by approximating the integrals that defined the coefficients of the the Fourier series:

$$\tilde{f}(\omega) = \frac{1}{\sqrt{L}} \sum_{j=0}^{N-1} e^{-2\pi i \omega x_j / L} f(x_j) \Delta x, \quad \omega = -N/2+1, \ldots, N/2. \tag{18}$$

Because the functions $e^{2\pi i \omega x/L}$ are orthogonal with respect to the discrete inner product

$$(u, v)_N = \Delta x \sum_{j=0}^{N-1} u(x_j) v(x_j), \tag{19}$$

it is straightforward to verify that $f_N(x)$ does in fact satisfy the conditions (17). Note that the discrete inner product is an approximation of the continuous inner product defined in (8).

From (17), we have

$$f(x_j) = \frac{1}{\sqrt{L}} \sum_{\eta=-N/2+1}^{N/2} e^{2\pi i \eta x_j / L} \tilde{f}(\eta). \tag{20}$$

Multiplying both sides by $\Delta x e^{-2\pi i \omega x_j / L}$, and summing from $j = 0$ to $j = N-1$ yields

$$\Delta x \sum_{j=0}^{N-1} e^{-2\pi i \omega x_j / L} f(x_j) = \Delta x \frac{1}{\sqrt{L}} \sum_{j=0}^{N-1} \sum_{\eta=-N/2+1}^{N/2} e^{-2\pi i \omega x_j / L} e^{2\pi i \eta x_j / L} \tilde{f}(\eta), \tag{21}$$

or

$$\Delta x \sum_{j=0}^{N-1} e^{-2\pi i \omega x_j / L} f(x_j) = \frac{1}{\sqrt{L}} \sum_{\eta=-N/2+1}^{N/2} \tilde{f}(\eta) \left[ \Delta x \sum_{j=0}^{N-1} e^{-2\pi i \omega x_j / L} e^{2\pi i \eta x_j / L} \right]. \tag{22}$$

Because

$$\left( e^{2\pi i \omega x/L}, e^{2\pi i \eta x/L} \right)_N = \left\{ \begin{array}{ll} L & \omega = \eta \\ 0 & \omega \neq \eta \end{array} \right. , \tag{23}$$

all terms in the outer sum on the right side of (22) vanish except for $\eta = \omega$, and we obtain the formula (18). It should be noted that the algebraic operations performed on (20) are equivalent to taking the discrete inner product of both sides of (20) with $e^{2\pi i \omega x/L}$.

The process of obtaining the approximate Fourier coefficients as in (18) is called the *discrete Fourier transform* (DFT) of $f(x)$. The discrete inverse Fourier transform is given by (16).

## 3 De-Noising and Aliasing

Suppose we have 128 data points sampled from the following function over $[0, 2\pi]$:

$$f(x) = \sin(10x) + \text{noise}. \tag{24}$$

The function $f(x)$, shown in Figure 1(a), is quite noisy. However, by taking the discrete Fourier transform, (Figure 1(b)) we can extract the original sine wave quite easily. The Fourier transform shows two distinct spikes, one at $n = 11$ and one at $n = 119$. These correspond to frequencies of $\pm 10$ i.e. the frequency of the original sine wave. The first $N/2 + 1$ values of the Fourier transform correspond to frequencies of $0 \leq f \leq f_{max}$, where $f_{max} = N/2$. The remaining $N/2 - 1$ values of the Fourier transform correspond to the frequencies $-f_{max} < f < 0$.

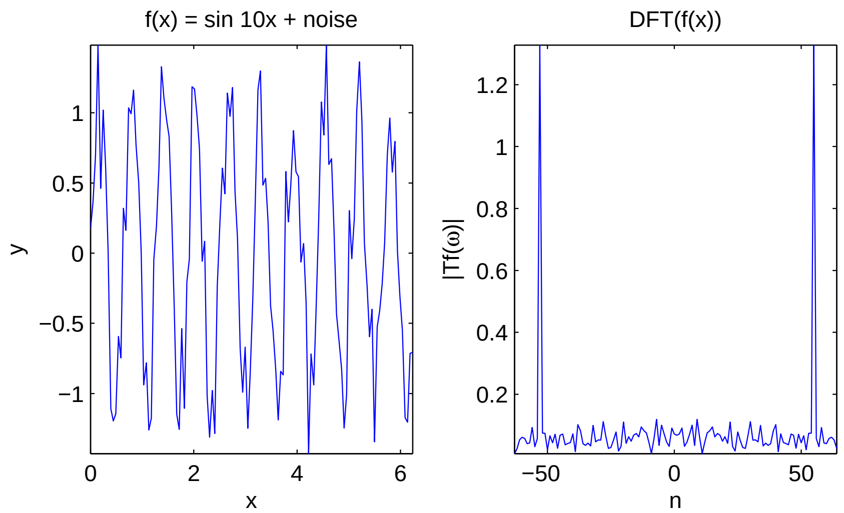


Figure 1: (a) Left plot: noisy signal (b) Right plot: discrete Fourier transform

Note that the DFT only considers a finite range of frequencies. If there are frequencies beyond this present in the true Fourier transform, an effect known as *aliasing* occurs. The effect of aliasing is shown in Figure 2: it "folds" these frequencies back into the computed Fourier transform. Specifically,

$$\tilde{f}(\omega) = \sum_{\ell=-\infty}^{\infty} \hat{f}(\omega + \ell N), \quad -N/2 + 1 \leq \omega \leq N/2. \tag{25}$$

Aliasing can be avoided by filtering the function before the DFT is applied, to prevent high-frequency components from "contaminating" the coefficients of the DFT.

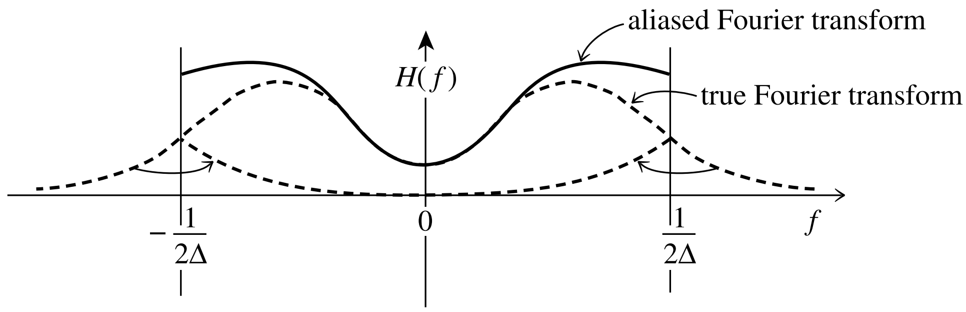


Figure 2: Aliasing effect (from "Numerical Recipes in C", Cambridge University Press)

## 4 Fast Fourier Transforms

The discrete Fourier transform, as it was presented in Section 2, requires $O(N^2)$ operations to compute. In fact the discrete Fourier transform can be computed much more efficiently than that ($O(N \log_2 N)$ operations) by using the fast Fourier transform (FFT).

The concept of the FFT is outlined below (based on "Numerical Recipes in C"). A more specialized text should be consulted for details of the implementation. The FFT arises by noting that a DFT of length $N$ can be written as the sum of two Fourier transforms each of length $N/2$. One of these transforms is formed from the even-numbered points of the original $N$, and the other from the odd-numbered points.

We have

$$\begin{aligned}
\tilde{f}(\omega) &= \frac{\Delta x}{\sqrt{L}} \sum_{j=0}^{N-1} e^{-2\pi i j\omega/N} f(x_j) \\
&= \frac{\Delta x}{\sqrt{L}} \sum_{j=0}^{N/2-1} e^{-2\pi i\omega(2j)/N} f(x_{2j}) + \frac{\Delta x}{\sqrt{L}} \sum_{j=0}^{N/2-1} e^{-2\pi i\omega(2j+1)/N} f(x_{2j+1}) \\
&= \frac{\Delta x}{\sqrt{L}} \sum_{j=0}^{N/2-1} e^{-2\pi i\omega j/(N/2)} f(x_{2j}) + \qquad (26)
\end{aligned}$$

$$\frac{\Delta x}{\sqrt{L}} W^{\omega} \sum_{j=0}^{N/2-1} e^{-2\pi i \omega j/(N/2)} f(x_{2j+1}) \tag{27}$$

where

$$W = e^{-2\pi i/N}. \tag{28}$$

It follows that

$$\tilde{f}(\omega) = \frac{1}{2}\tilde{f}^{e}(\omega) + \frac{1}{2}W^{\omega}\tilde{f}^{o}(\omega), \quad \omega = -N/2+1, \ldots, N/2, \tag{29}$$

where $\tilde{f}^{e}(\omega)$ is the DFT of $f$ obtained from its values at the even-numbered points of the $N$-point grid on which $f$ is defined, and $\tilde{f}^{o}(\omega)$ is the DFT of $f$ obtained from its values at the the odd-numbered points. Because the coefficients of a DFT of length $N$ are $N$-periodic, in view of the identity $e^{2\pi i} = 1$, evaluation of $\tilde{f}^{e}$ and $\tilde{f}^{o}$ at $\omega$ between $-N/2+1$ and $N/2$ is valid, even though they are transforms of length $N/2$ instead of $N$.

This reduction to half-size transforms can be performed recursively; i.e. a transform of length $N/2$ can be written as the sum of two transforms of length $N/4$, etc. Because only $O(N)$ operations are needed to construct a transform of length $N$ from two transforms of length $N/2$, the entire process requires only $O(N \log_2 N)$ operations.