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ENERGY 281
Spring Quarter 2007-08
Lecture 6 Notes

These notes are based on Rosalind Archer's PE281 lecture notes, with some revisions by Jim Lambers.

1 Higher-Dimensional Fourier Transforms

The Fourier transform generalizes naturally to higher dimensions. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ denote a point in $\mathbb{R}^n$. If a function $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is absolutely integrable on $\mathbb{R}^n$; that is, if

$$\int_{\mathbb{R}^n} |f(\mathbf{x})|\, d\mathbf{x} \tag{1}$$

exists and is finite, then its Fourier transform exists and is given by

$$\hat{f}(\mathbf{s}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\mathbf{s}\cdot\mathbf{x}} f(\mathbf{x})\, d\mathbf{x}, \tag{2}$$

where

$$\mathbf{s} \cdot \mathbf{x} = s_1 x_1 + s_2 x_2 + \cdots + s_n x_n \tag{3}$$

is the standard inner product of vectors in $\mathbb{R}^n$. The inverse transform is similarly generalized:

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\mathbf{s}\cdot\mathbf{x}} \hat{f}(\mathbf{s})\, d\mathbf{s}. \tag{4}$$

The DFT is generalized in the same way. It is implemented in Matlab by the `fftn` function, where the function `f` to which it is applied must be stored in an n-dimensional array. Then, `fftn` applies a one-dimensional `fft` to each dimension of the array. Along each dimension, the frequencies are ordered as in the one-dimensional case:

$$0, 1, 2, \ldots, N/2 - 1, N/2, -N/2 + 1, -N/2 + 2, \ldots, -2, -1 \tag{5}$$

where N is the number of points per dimension. If a function of n variables is represented as a vector with N^n elements, then the `reshape` function should be applied to it before `fftn` is used, or it will perform a one-dimensional `fft` on the entire vector instead.

2 The Laplace Transform

The *Laplace Transform* is defined by:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \hat{f}(s). \tag{6}$$

Example Let $f(t) = t$. Recalling integration by parts,

$$\int_a^b u \frac{dv}{dt}\, dt = uv|_a^b - \int_a^b v \frac{du}{dt}\, dt \tag{7}$$

Choose $u = t$ and $v = -\frac{1}{s}e^{-st}$. Then

$$\begin{aligned} \mathcal{L}\{f(t)\} &= -\frac{t}{s}e^{-st}\bigg|_0^\infty + \int_0^\infty \frac{1}{s}e^{-st}\, dt \\ &= 0 + \left[-\frac{1}{s^2}e^{-st}\right]_0^\infty \\ &= \frac{1}{s^2}. \end{aligned} \tag{8}$$

For the Laplace transform to exist, the following conditions must hold:

- $f(t)$ has a finite number of maxima, minima and discontinuities
- There exist constants α, M, and T such that

$$e^{-\alpha t}|f(t)| < M, \quad t > T. \tag{9}$$

Functions satisfying this requirement are known as functions of *exponential order*. For $t > 0$, there is $\alpha_1 > \alpha$ such that

$$e^{-\alpha_1 t}|f(t)| < M. \tag{10}$$

When this holds, α is known as the *abscissa of covergence*.

Example If $f(t) = e^{2t}$, then $e^{-\alpha t}f(t) = e^{-(\alpha-2)t}$ remains bounded for $\alpha \leq 2$. Therefore, the abscissa of convergence for $f(t)$ is 2.

The inverse Laplace transform, that computes $\mathcal{L}^{-1}\{\hat{f}(s)\} = f(t)$ from $\hat{f}(s)$, can be defined as an integral, but this definition is normally used for numerical inversion, rather than analytical. We will discuss this later.

3 Properties of Laplace Transforms

The following theorems facilitate the computation of Laplace and inverse Laplace transforms.

3.1 Theorem 1 - Linearity of the Laplace Transform

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\} \tag{11}$$

That is, the Laplace Transform is a linear operator.

Proof By the linearity of the integral,

$$\begin{aligned}
\mathcal{L}\{c_1 f_1 + c_2 f_2\} &= \int_0^\infty e^{-st}[c_1 f_1(t) + c_2 f_2(t)]\, dt \\
&= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t)\, dt \\
&= c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\}
\end{aligned} \tag{12}$$

3.2 Theorem 2 - Laplace Transform of a Derivative

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0). \tag{13}$$

Proof By definition,

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t) dt \tag{14}$$

Integrate by parts to obtain

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= e^{-st} f(t)\big|_0^\infty - \int_0^\infty f(t)(-se^{-st})\, dt \\
&= -f(0) + s\int_0^\infty e^{-st} f(t)\, dt \\
&= s\mathcal{L}\{f(t)\} - f(0).
\end{aligned} \tag{15}$$

3.3 Theorem 3 - Laplace Transform of Higher-Order Derivatives

$$\mathcal{L}\left\{\frac{\partial^n f}{\partial t^n}\right\} = s^n \mathcal{L}\{f(t)\} - \sum_{i=0}^{n-1} s^i f^{n-i-1}(0) \tag{16}$$

This can be proved by repeated application of Theorem 2.

3.4 Theorem 4 - Early Time Behavior

$$\lim_{s\to\infty} s\mathcal{L}\{f(t)\} = \lim_{t\to 0^+} f(t) = f(0^+) \tag{17}$$

Proof From Theorem 2,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^+) \tag{18}$$

Now take limits to obtain

$$\lim_{s\to\infty} \mathcal{L}\{f'(t)\} = \lim_{s\to\infty} s\mathcal{L}\{f(t)\} - f(0^+) \tag{19}$$

If $f'(t)$ is of exponential order, then

$$\lim_{s\to\infty} e^{-st} f'(t) = 0, \tag{20}$$

which means that for some constants M and α,

$$|f'(t)| < Me^{\alpha t}, \quad \forall t > 0. \tag{21}$$

Therefore

$$I(b) = \int_0^b |f'(t)| e^{-st}\, dt < \int_0^b Me^{\alpha t} e^{-st}\, dt = \left.\frac{Me^{-(s-\alpha)t}}{-(s-\alpha)}\right|_0^b \tag{22}$$

Because $s > \alpha$ as $b \to \infty$, we have

$$\lim_{b\to\infty} I(b) = \frac{M}{s-\alpha}, \tag{23}$$

so as $s \to \infty$, $I(b) \to 0$. It follows that the left hand side of (18) tends to zero; that is,

$$0 = \lim_{s\to\infty} s\mathcal{L}\{f(t)\} - f(0^+), \tag{24}$$

proving the theorem.

3.5 Theorem 5 - Late Time Behavior

$$\lim_{s\to 0} s\mathcal{L}\{f(t)\} = \lim_{t\to\infty} f(t) \tag{25}$$

Proof From Theorem 2,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^+). \tag{26}$$

Take limits to obtain

$$\lim_{s\to 0} \mathcal{L}\{f'(t)\} = \lim_{s\to 0} s\mathcal{L}\{f(t)\} - f(0^+). \tag{27}$$

Applying the definition of the transform to the left side, we obtain

$$\begin{aligned} \lim_{s\to 0} \int_0^\infty e^{-st} f'(t)\,dt &= \int_0^\infty f'(t) \lim_{s\to 0} e^{-st}\,dt \\ &= \int_0^\infty f'(t)\,dt \\ &= \lim_{t\to\infty} f(t) - f(0). \end{aligned} \tag{28}$$

Substituting (28) into the left side of (27) yields

$$\lim_{s\to 0} s\mathcal{L}\{f(t)\} = \lim_{t\to\infty} f(t). \tag{29}$$

3.6 Theorem 6 - Multiplication of a Transform by s

If $\mathcal{L}^{-1}\{\phi(s)\} = f(t)$, then

$$\mathcal{L}^{-1}\{s\phi(s)\} = f'(t). \tag{30}$$

Proof Let

$$f(t) = \mathcal{L}^{-1}\{\phi(s)\}, \quad g(t) = \mathcal{L}^{-1}\{s\phi(s)\}. \tag{31}$$

By Theorem 2,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{F(t)\} - f(0), \tag{32}$$

and by Theorem 4,

$$\begin{aligned} f(0) &= \lim_{s\to\infty} s\mathcal{L}\{f(t)\} \\ &= \lim_{s\to\infty} s\phi(s) \\ &= \lim_{s\to\infty} \mathcal{L}\{g(t)\} \\ &= 0. \end{aligned} \tag{33}$$

Now consider $f'(t)$ by returning to Theorem 2:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} = s\phi(s). \tag{34}$$

Taking inverse Laplace transforms, we obtain

$$\mathcal{L}^{-1}\{\mathcal{L}\{f'(t)\}\} = \mathcal{L}^{-1}\{s\phi(s)\}, \tag{35}$$

which simplifies to

$$f'(t) = \mathcal{L}^{-1}\{s\phi(s)\}. \tag{36}$$

From the definition of $f(t)$,

$$\frac{\partial}{\partial t}\mathcal{L}^{-1}\{\phi(s)\} = \mathcal{L}^{-1}\{s\phi(s)\}. \tag{37}$$

Example Suppose we want to find the inverse transform of:

$$\mathcal{L}\{f(t)\} = \frac{s}{s^2 + a^2} \tag{38}$$

We can use the following transform relationship to help us:

$$\mathcal{L}\left\{\frac{\sin at}{a}\right\} = \frac{1}{s^2 + a^2}. \tag{39}$$

Using Theorem 6, we obtain

$$f(t) = \frac{\partial}{\partial t}\left(\frac{\sin at}{a}\right) = \cos at. \tag{40}$$

3.7 Theorem 7 - Division of a Transform by s

If $\mathcal{L}^{-1}\{\phi(s)\} = f(t)$, then

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\phi(s)\right\} = \int_0^t f(\tau)\, d\tau. \tag{41}$$

Proof From the definition of the transform,

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)\, d\tau\right)\, dt \tag{42}$$

Using integration by parts, we obtain

$$-\left[\frac{1}{s}e^{-st}\int_0^t f(\tau)\, d\tau\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} f(t)\, dt = \frac{1}{s}\mathcal{L}\{f(t)\}. \tag{43}$$

Defining $\phi(s) = \mathcal{L}\{f(t)\}$ and taking inverse transforms yields the theorem.

Example Suppose we require the inverse transform of

$$\mathcal{L}\{f(t)\} = \frac{1}{s^3 + 4s} = \frac{1}{s}\frac{1}{s^2 + 4} \tag{44}$$

We know that

$$\mathcal{L}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t. \tag{45}$$

Therefore,

$$f(t) = \int_0^t \frac{1}{2}\sin 2t\,dt = \frac{1}{4}(1-\cos 2t). \tag{46}$$

3.8 Theorem 8 - First Shift Theorem

$$\mathcal{L}\{e^{-at}f(t)\} = \hat{f}(s+a). \tag{47}$$

Proof

$$\mathcal{L}\{e^{-at}f(t)\} = \int_0^\infty e^{-at}e^{-st}f(t)\,dt = \int_0^\infty e^{-(s+a)}f(t)\,dt = \hat{f}(s+a). \tag{48}$$

3.9 Theorem 9 - Second Shift Theorem

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}, \tag{49}$$

where $u(t-a)$ is a unit step function

$$u(t-a) = \left\{\begin{array}{ll} 1 & t-a>0 \\ 0 & \text{otherwise} \end{array}\right. . \tag{50}$$

Proof

$$\begin{aligned} \mathcal{L}\{f(t-a)u(t-a)\} &= \int_0^\infty f(t-a)u(t-a)e^{-st}\,dt \\ &= \int_a^\infty f(t-a)e^{-st}\,dt \\ &= \int_0^\infty f(\tau)e^{-s(\tau+a)}\,d\tau \end{aligned} \tag{51}$$

where $\tau = t-a$. Apply Theorem 8 to obtain

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-sa}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau = e^{-sa}\mathcal{L}\{f(t)\}. \tag{52}$$

3.10 Theorem 10 - Multiplication by t

$$\mathcal{L}\{tf(t)\} = -\hat{f}'(s) \tag{53}$$

Proof

$$\hat{f}'(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty -te^{-st}f(t)\,dt = -\mathcal{L}\{tf(t)\}. \tag{54}$$

3.11 Theorem 11 - Division by t

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \hat{f}(s)\,ds \tag{55}$$

Proof

$$\begin{aligned}
\int_s^{\infty} \hat{f}(s)\,ds &= \int_s^{\infty}\int_0^{\infty} e^{-st} f(t)\,dt\,ds \\
&= \int_0^{\infty}\int_s^{\infty} e^{-st} f(t)\,ds\,dt \\
&= \int_0^{\infty} f(t)\left[\frac{e^{-st}}{-t}\right]_s^{\infty} dt \\
&= \int_0^{\infty} \frac{f(t)}{t} e^{-st}\,dt \\
&= \mathcal{L}\left\{\frac{f(t)}{t}\right\}.
\end{aligned} \tag{56}$$

3.12 Theorem 12 - Convolution

$$\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \mathcal{L}\left\{\int_0^t f(t-\lambda)g(\lambda)\,d\lambda\right\} \tag{57}$$

$$= \mathcal{L}\left\{\int_0^t f(\lambda)g(t-\lambda)\,d\lambda\right\} \tag{58}$$

$$= \mathcal{L}\{f(t) * g(t)\}. \tag{59}$$

Proof First, use the definition of the Laplace transform to obtain

$$\mathcal{L}\left\{\int_0^t f(t-\lambda)g(\lambda)\,d\lambda\right\} = \int_0^{\infty}\int_0^t f(t-\lambda)g(\lambda)e^{-st}\,d\lambda\,dt. \tag{60}$$

Change limits on the λ integral by introducing a step function:

$$\mathcal{L}\left\{\int_0^t f(t-\lambda)g(\lambda)\,d\lambda\right\} = \int_0^{\infty}\int_0^{\infty} u(t-\lambda)f(t-\lambda)g(\lambda)e^{-st}\,d\lambda\,dt. \tag{61}$$

Changing the order of integration yields

$$\mathcal{L}\left\{\int_0^t f(t-\lambda)g(\lambda)\,d\lambda\right\} = \int_0^{\infty} g(\lambda)\int_0^{\infty} u(t-\lambda)f(t-\lambda)e^{-st}\,dt\,d\lambda. \tag{62}$$

Take the step function into account:

$$\mathcal{L}\left\{\int_0^t f(t-\lambda)g(\lambda)\,d\lambda\right\} = \int_0^\infty g(\lambda)\int_\lambda^\infty f(t-\lambda)e^{-st}\,dt\,d\lambda \tag{63}$$

Apply first shift theorem:

$$\begin{aligned}\mathcal{L}\left\{\int_0^t f(t-\lambda)g(\lambda)\,d\lambda\right\} &= \int_0^\infty g(\lambda)\left[\int_0^\infty f(\tau)e^{-s(\tau+\lambda)}\,d\tau\right]d\lambda \\ &= \int_0^\infty g(\lambda)e^{-s\lambda}\,d\lambda\int_0^\infty f(\tau)e^{-s\tau}\,d\tau \\ &= \mathcal{L}\{g(t)\}\mathcal{L}\{f(t)\}, \end{aligned} \tag{64}$$

where $\tau = t - \lambda$.

4 Solving Differential Equations with Laplace Transforms

Laplace transforms can be used as a powerful tool to solve differential equations. The general procedure is:

1. transform both sides of the equation
2. solve the transformed equation to get an expression for the Laplace transform of the solution
3. invert to find the solution in real space

This approach turns an ordinary differential equation into an alegbraic equation and a partial differential equation in x and t into an ordinary differential equation in x or t.

4.1 Ordinary Differential Equation Example

We solve

$$y'' + 2y' + y = te^{-t}, \tag{65}$$

with initial conditions

$$y(0) = 1, \quad y'(0) = -2. \tag{66}$$

From Theorem 3, we have

$$\mathcal{L}\{y''(t)\} = s^2\hat{y} - sy(0) - y'(0), \quad \mathcal{L}\{y'(t)\} = s\hat{y} - y(0), \tag{67}$$

and by Theorem 8,

$$\mathcal{L}\{te^{-t}\} = \frac{1}{(s+1)^2}. \tag{68}$$

It follows that

$$s^2\hat{y}(s) - s + 2 + 2s\hat{y}(s) - 2 + \hat{y}(s) = \frac{1}{(s+1)^2}. \tag{69}$$

Now, solve this algebraic equation for $\hat{y}$:

$$(s^2 + 2s + 1)\hat{y}(s) - s = \frac{1}{(s+1)^2} \tag{70}$$

$$\implies \quad (s+1)^2\hat{y}(s) = \frac{1}{(s+1)^2} + s \tag{71}$$

$$\implies \quad \hat{y}(s) = \frac{1}{(s+1)^4} + \frac{s}{(s+1)^2}. \tag{72}$$

Now, invert the transform to find $y(t)$. For the first term, we use a table to obtain

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}. \tag{73}$$

Combining this transform with the first shift theorem gives

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^4}\right\} = \frac{e^{-t}t^3}{3!}. \tag{74}$$

Now consider the second term:

$$\frac{s}{(s+1)^2} = \frac{s+1-1}{(s+1)^2} = \frac{s+1}{(s+1)^2} - \frac{1}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2}. \tag{75}$$

We can use the known transforms

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t\} = \frac{1}{s^2}. \tag{76}$$

Combining this with the first shift theorem again gives

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t} - e^{-t}t. \tag{77}$$

The final solution is

$$y(t) = e^{-t}\left(\frac{t^3}{3!} - t + 1\right). \tag{78}$$