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Lecture 7 Notes

These notes are based on Rosalind Archer's PE281 lecture notes, with some revisions by Jim Lambers.

Petroleum Engineering Applications of Laplace Transforms

This lecture and the next outline how Laplace transforms can be used to solve problems of interest to petroleum engineers. The solutions presented consider different treatments of the well and different boundary conditions.

1 Line Source Solution

This section considers infinite acting radial flow in a reservoir where the well is modeled as a line source. The differential equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = \frac{\phi\mu c_t}{k}\frac{\partial p}{\partial t}. \tag{1}$$

A constant rate boundary condition is specified at $r = 0$:

$$\lim_{r\to 0}\frac{2\pi kh}{\mu}r\frac{\partial p}{\partial r} = q. \tag{2}$$

The outer boundary condition is

$$\lim_{r\to\infty} p(r,t) = p_i, \quad t > 0. \tag{3}$$

The initial condition is

$$p(r,0) = p_i, \quad r > 0. \tag{4}$$

This can be written in dimensionless form as follows:

$$p_D = \frac{2\pi kh}{q\mu}(p_i - p), \tag{5}$$

$$r_D = \frac{r}{r_w}, \tag{6}$$

$$t_D = \frac{kt}{\phi\mu c_t r_w^2}. \tag{7}$$

The resulting differential equation and boundary conditions are

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial p_D}{\partial r_D}\right) = \frac{\partial p_D}{\partial t_D}, \tag{8}$$

$$p_D(r_D, 0) = 0, \tag{9}$$

$$\lim_{r_D\to\infty} p_D(r_D, t_D) = 0, \tag{10}$$

$$\lim_{r_D\to 0} r_D\frac{\partial p_D}{\partial t_D} = -1. \tag{11}$$

The solution procedure begins by taking the Laplace transfor of both sides of the pressure equation:

$$\mathcal{L}\left\{\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial p_D}{\partial r_D}\right)\right\} = \mathcal{L}\left\{\frac{\partial p_D}{\partial t_D}\right\} \tag{12}$$

$$\implies \quad \frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial \hat{p}_D}{\partial r_D}\right) = s\hat{p}_D(r_D, s) - p_D(r_D, 0) \tag{13}$$

$$\implies \quad \frac{\partial^2 \hat{p}_D}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial \hat{p}_D}{\partial r_D} - s\hat{p}_D(r_D, s) = 0 \tag{14}$$

A solution to this differential equation can be found by noting that it is an example of a modified Bessel equation.

1.1 Bessel and Modified Bessel Equations

The *Bessel equation* is

$$x^2y'' + xy' + (x^2 - n^2)y = 0. \tag{15}$$

The general solution of this equation has the form

$$y(x) = c_1J_n(x) + c_2Y_n(x), \tag{16}$$

where J_n is a *Bessel function of the first kind* of order n and Y_n is a *Bessel function of the second kind* of order n.

The modified Bessel equation is

$$x^2y'' + xy' - (x^2 + n^2)y = 0. \tag{17}$$

The general solution of this equation has the form

$$y(x) = c_1I_n(x) + c_2K_n(x), \tag{18}$$

where I_n and K_n are *modified Bessel functions* of the first and second kind, respectively, of order n.

1.2 Laplace Space Solution for p_D

The transformed pressure equation can be written as

$$r_D^2 \frac{\partial^2 \hat{p}_D}{\partial r_D^2} + r_D \frac{\partial \hat{p}_D}{\partial r_D} - r_D^2 s \hat{p}_D = 0. \tag{19}$$

Substitute $\xi = r_D \sqrt{s}$ to obtain

$$\xi^2 \frac{\partial^2 \hat{p}_D}{\partial \xi^2} + \xi \frac{\partial \hat{p}_D}{\partial \xi} - \xi^2 \hat{p}_D = 0. \tag{20}$$

Now, the equation is the modified Bessel equation of order zero, and we can solve for $\hat{p}_D$ to obtain the general solution

$$\hat{p}_D(r_D, s) = c_1(s) I_0(r_D \sqrt{s}) + c_2(s) K_0(r_D \sqrt{s}). \tag{21}$$

Now consider the boundary conditions, beginning with the infinite-acting condition. As $r_D \to \infty$, $\hat{p}_D$ must remain bounded. However,

$$\lim_{x \to \infty} I_0(x) = \infty. \tag{22}$$

To prevent $\hat{p}_D$ from going to infinity we set $c_1(s) = 0$, which leaves

$$\hat{p}_D(r_D, s) = c_2(s) K_0(r_D \sqrt{s}). \tag{23}$$

Recall that the inner boundary condition is

$$\lim_{r_D \to 0} r_D \frac{\partial p_D}{\partial r_D} = -1. \tag{24}$$

Taking the Laplace transform of both sides yields

$$\lim_{r_D \to 0} \mathcal{L} \left\{ r_D \frac{\partial p_D}{\partial r_D} \right\} = \lim_{r_D \to 0} r_D \frac{\partial \hat{p}_D}{\partial r_D} = \mathcal{L}\{-1\} = -\frac{1}{s}. \tag{25}$$

To differentiate the Bessel function we need the following recurrence relation:

$$\frac{d}{dx} \left(x^{-n} K_n(x) \right) = -x^{-n} K_{n+1}(x) \tag{26}$$

Substituting (23) into (25) gives:

$$\lim_{r_D \to 0} r_D \frac{\partial}{\partial r_D} \left(c_2(s) K_0(r_D \sqrt{s}) \right) = -\frac{1}{s}. \tag{27}$$

Then, by (26),

$$\lim_{r_D\to 0}\left[-c_2(s)r_D\sqrt{s}K_1(r_D\sqrt{s})\right] = -\frac{1}{s}. \tag{28}$$

To evaluate the limit we can use the following limiting form of K_v for small arguments:

$$K_v(z) \approx \frac{1}{2}\Gamma(v)\left(\frac{1}{2}z\right)^{-v} \tag{29}$$

where $\Gamma(v)$ is the gamma function. It follows that

$$\lim_{r_D\to 0} K_1(r_D\sqrt{s}) = \frac{1}{r_D\sqrt{s}}. \tag{30}$$

Substituting into (28) yields

$$-c_2(s)\lim_{r_D\to 0}\left(r_D\sqrt{s}\frac{1}{r_D\sqrt{s}}\right) = -\frac{1}{s} \tag{31}$$

and we conclude that

$$c_2(s) = \frac{1}{s}. \tag{32}$$

Finally, we have the complete solution for $\hat{p}_D$:

$$\hat{p}_D(r_D, s) = \frac{1}{s}K_0(r_D\sqrt{s}). \tag{33}$$

Now invert the Laplace transform to find p_D. This can be achieved by recalling Theorem 7:

$$\mathcal{L}^{-1}\{\phi(s)\} = f(t) \quad\Longrightarrow\quad \mathcal{L}^{-1}\left\{\frac{1}{s}\phi(s)\right\} = \int_0^t f(\tau)\,d\tau. \tag{34}$$

To proceed, the inverse transform of $K_0(r_S\sqrt{s})$ is required. From a table, we obtain

$$\mathcal{L}^{-1}\{K_0(r_D\sqrt{s})\} = \frac{1}{2t_D}\exp\left(\frac{-r_D^2}{4t_D}\right) \tag{35}$$

and by Theorem 7,

$$p_D(r_D, t_D) = \int_0^{t_D}\frac{1}{2\tau}\exp\left(\frac{-r_D^2}{4\tau}\right)\,d\tau. \tag{36}$$

This integral can be evaluated by using the substitution

$$u = \frac{r_D^2}{4\tau} \tag{37}$$

which yields the relations

$$\tau = \frac{r_D^2}{4u}, \quad d\tau = \frac{-r_D^2}{4u^2}du. \tag{38}$$

The integral (36) becomes

$$p_D(r_D, t_D) = \int_{\infty}^{\frac{r_D^2}{4t_D}} \frac{4u}{2r_D^2} \exp(-u) \frac{-r_D^2}{4u^2}\, du = \frac{1}{2} \int_{\frac{r_D^2}{4t_D}}^{\infty} \frac{\exp(-u)}{u}\, du. \tag{39}$$

Now introduce the *exponential integral*, denoted by $Ei(x)$:

$$Ei(x) = -\int_{-x}^{\infty} \frac{e^{-u}}{u}\, du. \tag{40}$$

This definition follows Abramowitz and Stegun, "Handbook of Mathematical Functions", Dover, 1970. Note that in some references, $Ei(x)$ is denoted by $E_1(x)$, so be careful! Using this function, we obtain

$$p_D(r_D, t_D) = -\frac{1}{2} Ei\left(-\frac{r_D^2}{4t_D}\right). \tag{41}$$

Finally, the solution is written in dimensional terms:

$$p(r, t) = p_i + \frac{q\mu}{4\pi kh} Ei\left(-\frac{r^2 \phi \mu c_t}{4kt}\right). \tag{42}$$

1.3 Late Time Behavior of p_D

We can consider the late time behavior of p_D by examining the behavior of $\hat{p}_D$ as $s \to 0$. Recall that

$$\hat{p}_D(r_D, s) = \frac{1}{s} K_0(r_D \sqrt{s}). \tag{43}$$

The limit can be handled by using a series expansion for K_0:

$$K_0(x) = -\left(\ln \frac{x}{2} + \gamma\right) I_0(x) + \frac{\frac{1}{4}z^2}{(1!)^2} + \left(1 + \frac{1}{2}\right) \frac{(\frac{1}{4}z^2)^2}{(2!)^2} + ... \tag{44}$$

$$\implies \lim_{x \to 0} K_0(x) = -\ln\left(\frac{x}{2}\right) - \gamma \tag{45}$$

where γ = Euler's constant, 0.5772. Using a table of Laplace transforms, we obtain

$$\begin{aligned} p_D(r_D, t_D) &= \mathcal{L}^{-1}\{\hat{p}_D(r_D, s)\} \\ &\approx \mathcal{L}^{-1}\left\{-\frac{1}{s}(\ln r_D + \ln\sqrt{s} - \ln 2 + \gamma)\right\} \\ &= -\ln r_D + \ln 2 - \gamma + \frac{1}{2}(\gamma + \ln t_D) \\ &= \frac{1}{2}\left(\ln\frac{t_D}{r_D^2} + 0.80907\right). \end{aligned} \tag{46}$$

2 Finite Well Radius Solution

The line source solution applies the constant flow rate condition as r tends to zero, which simplifies the solution process. However, an analytical solution can also be obtained when the flow rate condition is applied at $r = r_w$. The governing equation and boundary conditions remain the same as with the line source solution, except for the inner boundary condition which is now

$$r_D \frac{\partial p_D}{\partial r_D}\bigg|_{r_D=1} = -1. \tag{47}$$

As before, when the differential equation is transformed, we have

$$\frac{\partial^2 \hat{p}_D}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial \hat{p}_D}{\partial r_D} - s\hat{p}_D = 0 \tag{48}$$

which has the same general solution

$$\hat{p}_D(r_D, s) = c_1(s) I_0(r_D\sqrt{s}) + c_2(s) K_0(r_D\sqrt{s}). \tag{49}$$

The solution must remain bounded as $r_D \to \infty$, so we again set $c_1(s) = 0$:

$$\hat{p_D}(r_D, s) = c_2(s) K_0(r_D\sqrt{s}). \tag{50}$$

Now, consider the inner boundary condition. It requires that

$$\frac{\partial}{\partial r_D}\left(c_2(s) K_0(r_D\sqrt{s})\right) = -\frac{1}{s} \tag{51}$$

From (26),

$$-c_2(s)\sqrt{s} K_1(r_D\sqrt{s}) = -\frac{1}{s}. \tag{52}$$

Evaluating at $r_D = 1$ and solving for $c_2(s)$ yields

$$c_2(s) = \frac{1}{s^{\frac{3}{2}} K_1(\sqrt{s})}. \tag{53}$$

Therefore, the final solution for $\hat{p}_D$ is

$$\hat{p}_D(r_D, s) = \frac{K_0(r_D\sqrt{s})}{s^{\frac{3}{2}} K_1(\sqrt{s})}. \tag{54}$$

2.1 Early Time Behavior of p_D

The early time behavior of $p_D(r_D, t_D)$ can be examined by considering the limit of $\hat{p}_D(r_D, s)$ as $s \to \infty$. To do so, the behavior of the Bessel functions for large arguments is required. We have

$$K_v(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x} \tag{55}$$

where x is large. Setting $v = 0$ yields

$$K_0(r_D\sqrt{s}) = \sqrt{\frac{\pi}{2r_D\sqrt{s}}} e^{-r_D\sqrt{s}}, \tag{56}$$

and setting $v = 1$ and $r_D = 1$ yields

$$K_1(\sqrt{s}) = \sqrt{\frac{\pi}{2\sqrt{s}}} e^{-\sqrt{s}}. \tag{57}$$

From (54), we see that at late time,

$$\begin{aligned} \hat{p}_D(r_D, s) &\approx \frac{1}{s^{\frac{3}{2}}} \sqrt{\frac{\pi}{2r_D\sqrt{s}}} e^{-r_D\sqrt{s}} \sqrt{\frac{2\sqrt{s}}{\pi}} e^{\sqrt{s}} \\ &= \frac{1}{s^{\frac{3}{2}}} \frac{1}{\sqrt{r_D}} e^{-\sqrt{s}(r_D - 1)} \end{aligned} \tag{58}$$

The early time behavior of p_D can be found using the `invlaplace` function in Maple:

$$p_D(r_D, t_D) \approx \frac{1}{\sqrt{r_D}} \left\{ 2\sqrt{\frac{t_D}{\pi}} e^{-\frac{(r_D-1)^2}{4t_D}} - (r_D - 1)\mathrm{erfc}\left(\frac{r_D - 1}{2\sqrt{t_D}}\right) \right\}. \tag{59}$$

It follows that at the well, the early time behavior is

$$p_D(1, t_D) = 2\sqrt{\frac{t_D}{\pi}}. \tag{60}$$

2.2 Late Time Behavior of p_D

To find the late time behavior of p_D consider the limit of $\hat{p}_D$ at $s \to 0$. First, consider the behavior of the Bessel functions. As before,

$$K_0(x) \approx -\left[\ln\left(\frac{1}{2}x\right) + \gamma\right] \tag{61}$$

For small arguments, $K_v(x)$ can be approximated by

$$K_v(x) \approx \frac{1}{2}\Gamma(v)\left(\frac{1}{2}x\right)^{-v}, \tag{62}$$

where $\Gamma(v) = (v-1)!$ Substituting $v = 1$ yields

$$K_1(x) \approx \frac{1}{2}\left(\frac{1}{2}x\right)^{-1} = \frac{1}{x}. \tag{63}$$

Recall the solution for $\hat{p}_D$:

$$\hat{p}_D(r_D, s) = \frac{K_0(r_D\sqrt{s})}{s^{\frac{3}{2}}K_1(\sqrt{s})}. \tag{64}$$

The late time behavior of p_D can be found by applying (61) and (63):

$$\begin{aligned} \hat{p}_D(r_D, s) &= \frac{K_0(r_D\sqrt{s})}{s^{\frac{3}{2}}K_1(\sqrt{s})} \\ &\approx \frac{-[\ln(\frac{1}{2}r_D\sqrt{s}) + \gamma]}{s^{\frac{3}{2}}\frac{1}{\sqrt{s}}} \\ &= \frac{-[\ln(\frac{1}{2}r_D\sqrt{s}) + \gamma]}{s}. \end{aligned} \tag{65}$$

Using the `invlaplace` function in Maple yields

$$\begin{aligned} p_D(r_D, t_D) &= \mathcal{L}^{-1}\left\{\frac{-[\ln(\frac{1}{2}r_D\sqrt{s}) + \gamma]}{s}\right\} \\ &= \mathcal{L}^{-1}\left\{-\frac{1}{s}(\ln r_D + \ln\sqrt{s} - \ln 2 + \gamma)\right\} \\ &= \frac{1}{2}\left(\ln\frac{t_D}{r_D^2} + 0.80907\right). \end{aligned} \tag{66}$$

This is the same late time behavior as the line source solution.