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ENERGY 281
Spring Quarter 2007-08
Midterm Exam Solution

1. Use separation of variables to solve the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2\pi, \quad t > 0,$$

with initial condition

$$u(x, 0) = 2\cos 2x - 4\cos 3x, \quad 0 < x < 2\pi,$$

and boundary conditions

$$u_x(0, t) = u_x(2\pi, t) = 0, \quad t > 0.$$

Solution We assume that the solution $u(x, t)$ has the form

$$u(x, t) = \sum_{k=0}^{\infty} a_k u_k(x, t),$$

where each a_k is a constant and each function $u_k(x, t)$ has the form

$$u_k(x, t) = M_k(x)N_k(t).$$

Because the PDE is linear, each function $u_k(x, t)$ must be a solution of the PDE, and satisfy the boundary conditions. Substituting $u_k(x, t)$ into the PDE yields

$$M_k(x)N_k'(t) = M_k''(x)N_k(t),$$

and subtituting into the boundary conditions yields

$$M_k'(0)N_k(t) = M_k'(2\pi)N_k(t) = 0.$$

If we assume that $M_k(x)$ and $N_k(t)$ are nonzero, then we obtain the conditions

$$\frac{N_k'(t)}{N_k(t)} = \frac{M_k''(x)}{M_k(x)}, \tag{1}$$

and

$$M_k'(0) = M_k'(2\pi) = 0.$$

Because the left side of (1) depends on t and the right side depends on x, but they must be equal for all x and t, they must equal a constant, which we denote by $-\lambda_k$. This yields the ODEs

$$M_k''(x) + \lambda_k M_k(x) = 0,$$

$$N_k'(t) + \lambda_k N_k(t) = 0.$$

The first equation has the general solution

$$M_k(x) = A_k \cos\sqrt{\lambda_k}x + B_k \sin\sqrt{\lambda_k}x.$$

From the boundary conditions, we obtain

$$B_k = 0, \quad 2\pi\sqrt{\lambda_k} = k\pi$$

for each integer k. It follows that

$$M_k(x) = \cos\frac{kx}{2}, \quad \lambda_k = \left(\frac{k}{2}\right)^2,$$

where we set $A_k = 1$ for convenience. Now, we have

$$N_k(t) = C_k e^{-\lambda_k t} = e^{-(k/2)^2 t},$$

where we set $C_k = 1$ for convenience. It follows that

$$u(x,t) = \sum_{k=0}^{\infty} a_k e^{-(k/2)^2 t} \cos\frac{kx}{2},$$

where a_k is chosen to satisfy the initial condition. Setting $t = 0$ and taking the inner product with $\cos(jx/2)$ yields

$$\left(\cos\frac{jx}{2}, 2\cos 2x - 4\cos 3x\right) = \sum_{k=0}^{\infty} a_k \left(\cos\frac{jx}{2}, \cos\frac{kx}{2}\right).$$

From the orthogonality of the cosines, we obtain $a_4 = 2$, $a_6 = -4$, and $a_k = 0$ for all other values of k. We conclude that the solution is

$$u(x,t) = 2e^{-4t}\cos 2x - 4e^{-9t}\cos 3x.$$

2. Use a Fourier sine or cosine transform to solve the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad t > 0,$$

with initial condition

$$u(x,0) = \delta(x), \quad x > 0,$$

where $\delta(x)$ is the Dirac delta function, and boundary condition

$$u_x(0,t) = \cos t, \quad t > 0.$$

Your final answer may include integrals with respect to time.

Hints:

- The Fourier cosine transform of an even function, or the Fourier sine function of an odd function, is the same as its full Fourier transform.
- The Dirac delta function $\delta(x)$ is an even function.

Solution Since the problem specifies a Neumann boundary condition, we must use the Fourier cosine transform, which yields the ODE

$$\hat{u}_t + s^2\hat{u} = -\sqrt{\frac{2}{\pi}}\cos t, \quad \hat{u}(s,0) = \frac{1}{\sqrt{2\pi}}.$$

Here we have used the fact that the Fourier cosine transform of an even function, namely $\delta(x)$, is the same as its full Fourier transform. Using the integrating factor $e^{s^2 t}$, we obtain

$$\hat{u}(s,t) = e^{-s^2 t}\frac{1}{\sqrt{2\pi}} - \sqrt{\frac{2}{\pi}}\int_0^t e^{-s^2(t-\tau)}\cos\tau\, d\tau.$$

Inverting the transform yields

$$\begin{aligned} u(x,t) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{u}(s,t)\cos sx\, ds \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \left[e^{-s^2 t}\frac{1}{\sqrt{2\pi}} - \sqrt{\frac{2}{\pi}}\int_0^t e^{-s^2(t-\tau)}\cos\tau\, d\tau\right]\cos sx\, ds \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-s^2 t}\cos sx\, ds - \\
&\quad \sqrt{\frac{2}{\pi}}\int_0^t \left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-s^2(t-\tau)}\cos sx\, ds\right]\cos\tau d\tau \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2t}}e^{-x^2/4t} - \sqrt{\frac{2}{\pi}}\int_0^t \frac{1}{\sqrt{2(t-\tau)}}e^{-x^2/4(t-\tau)}\cos\tau\, d\tau
\end{aligned}
$$

3. Use the Laplace transform to solve the ODE

$$y'' + 4y' + 13y = 1,$$

with initial conditions

$$y(0) = 1, \quad y'(0) = -2.$$

Your final solution may include an integral with respect to time.

Solution Applying the rules for transforming derivatives yields

$$s^2\hat{y} - sy(0) - y'(0) + 4(s\hat{y} - y(0)) + 13\hat{y} = \frac{1}{s}$$

which simplifies to

$$(s^2 + 4s + 13)\hat{y} - (4 + s)y(0) - y'(0) = \frac{1}{s}.$$

Solving for $\hat{y}(s)$ yields

$$\hat{y}(s) = \frac{\frac{1}{s} + (4 + s)y(0) + y'(0)}{(s+2)^2 + 9}$$

or, upon applying the initial conditions,

$$\hat{y}(s) = \frac{1}{s}\frac{1}{(s+2)^2 + 9} + \frac{s+2}{(s+2)^2 + 9}.$$

Inverting the transform yields

$$y(t) = \frac{1}{3}\int_0^t e^{-2\tau}\sin 3\tau\, d\tau + e^{-2t}\cos 3t.$$

This is acceptable as a final answer.

Applying integration by parts yields

$$\begin{aligned}
\int_0^t e^{-2\tau}\sin 3\tau\,d\tau &= -\frac{1}{3}e^{-2\tau}\cos 3\tau\Big|_0^t - \frac{2}{3}\int_0^t e^{-2\tau}\cos 3\tau\,d\tau \\
&= -\frac{1}{3}e^{-2\tau}\cos 3\tau\Big|_0^t - \\
&\quad \frac{2}{3}\left[\frac{1}{3}e^{-2\tau}\sin 3\tau\Big|_0^t + \frac{2}{3}\int_0^t e^{-2\tau}\sin 3\tau\,d\tau\right] \\
&= \frac{9}{13}\left[-\frac{1}{3}e^{-2\tau}\cos 3\tau - \frac{2}{9}e^{-2\tau}\sin 3\tau\right]\Big|_0^t \\
&= \frac{3}{13} - \frac{3}{13}e^{-2t}\cos 3t - \frac{2}{13}e^{-2t}\sin 3t.
\end{aligned}$$

We conclude that

$$y(t) = \frac{1}{13} + \frac{12}{13}e^{-2t}\cos 3t - \frac{2}{39}e^{-2t}\sin 3t.$$

4. Consider the function

$$f(x) = \cos 16x + \cos 144x$$

on the interval $[0, 2\pi]$.

(a) Compute the coefficients of the Fourier series for this function, using the complex exponential form.

Solution The coefficients are

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-i\omega x} f(x)\,dx.$$

Writing $f(x)$ as

$$f(x) = \frac{1}{2}\left[e^{16ix} + e^{-16ix} + e^{144ix} + e^{-144ix}\right],$$

and using the orthogonality of $e^{i\omega x}$, we obtain

$$\hat{f}(16) = \hat{f}(-16) = \hat{f}(144) = \hat{f}(-144) = \sqrt{\frac{\pi}{2}},$$

and $\hat{f}(\omega) = 0$ for all other values of ω.

(b) Compute the discrete Fourier transform of this function, on a 128-point grid. How do the coefficients differ from those in part 4a, and why?

Solution The coefficients of the DFT are related to the coefficients of the Fourier series by

$$\tilde{f}(\omega) = \sum_{\ell=-\infty}^{\infty} \hat{f}(\omega + \ell N), \quad \omega = -N/2+1, \ldots, N/2,$$

where $N = 128$. Since 144 is outside of the range $-63, \ldots, 64$, it follows that $\hat{f}(144)$ is included in $\tilde{f}(16)$, and $\hat{f}(-144)$ is included in $\tilde{f}(-16)$. That is, the coefficients of the DFT are

$$\tilde{f}(16) = \sqrt{2\pi}, \quad \tilde{f}(-16) = \sqrt{2\pi},$$

and $\tilde{f}(\omega) = 0$ for all other values of ω.