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ENERGY 281
Spring Quarter 2006-07
Practice Final Exam

1. Free Space Green's Functions

   The two-dimensional Helmholtz operator satisfies the equation

$$L(u) = \nabla^2 u + k^2 u \tag{1}$$

   where k is a constant. Using $\nabla^2 w + k^2 w = \delta(\xi - x, \eta - y)$, find the free space Green's function for the Helmholtz operator.

   (a) Show that the Helmholtz operator $L = \nabla^2 + k^2 I$ is self-adjoint.
   Solution Applying the Gauss divergence theorem twice gives

$$\begin{aligned}
(G, L(u)) &= \int_\Omega G[\nabla^2 u + k^2 u]\, d\Omega \\
&= \int_{\partial\Omega} \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n} \right) ds + \\
&\quad \int_\Omega u\nabla^2 G + k^2 uG\, d\Omega \\
&= (u, L(G)) + \text{boundary terms}
\end{aligned}$$

   (b) Write the Helmholtz operator in radial coordinates around the singular point (ξ, η). Include the boundary condition at $r = 0$.
   Solution The free space Green's function satisfies the equation

$$\frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} + k^2 w = \delta(\xi - x, \eta - y)$$

   In polar coordinates, this equation becomes

$$\begin{gathered}
\tfrac{1}{r}\tfrac{\partial}{\partial r}\left(r\tfrac{\partial w}{\partial r}\right) + k^2 w(r) = 0 \\
\lim_{r \to 0} w(r) = \infty
\end{gathered}$$

   Simplifying yields

$$\frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial r^2} + k^2 w = 0.$$

(c) Use a change of variables to write the resulting ODE as a Bessel's Equation and find the general solution.

Solution The correct change of variables is $s = kr$. Applying this change yields

$$s^2 \frac{\partial^2 w}{\partial s^2} + s \frac{\partial w}{\partial s} + s^2 w = 0$$

This is Bessel's equation of order zero, which has the general solution

$$w(r) = AJ_0(s) + BY_0(s) = AJ_0(kr) + BY_0(kr)$$

where A and B are constants.

(d) The solution for the boundary condition is $\frac{1}{4}Y_0\,(kr)$. (You do not have to find this). Verify that this solution satisfies the condition that w must blow up at (ξ, η).

Solution By definition $Y_0(s)$ blows up as $s \to 0$. As $x \to \xi$ and $y \to \eta$, the argument $s = kr \to 0$, so the solution blows up as required.

2. Method of Images

Solve the Helmholtz problem for the following boundary conditions:

$$\nabla^2 u + k^2 u = 0 \tag{2}$$

$$y > 0 \tag{3}$$

$$u\,(x, 0) = f\,(x) \tag{4}$$

(a) Write the equation for $u\,(\xi, \eta)$ in terms of boundary integrals.

Solution Because the boundary consists of the line $y = 0$, we have

$$u(\xi, \eta) = -\int_{-\infty}^{\infty} \left(G(s, 0; \xi, \eta) \frac{\partial u(s, 0)}{\partial n} - u(s, 0) \frac{\partial G(s, 0; \xi, \eta)}{\partial n} \right) ds$$

where $\frac{\partial}{\partial n} = -\frac{\partial}{\partial y}$.

(b) What assumptions do you need to make about the Green's function G on the boundary in order to get a problem that can be solved?

Solution You must assume that $G(x, y; \xi, \eta) = 0$ on the line $y = 0$ because we don't know the normal derivative of u on the boundary.

(c) Find the Green's function $G = w + g$ for these boundary conditions.

Solution Using the method of images the solution is

$$\frac{1}{4}Y_0\left(k\sqrt{(\xi-x)^2+(\eta+y)^2}\right)-\frac{1}{4}Y_0\left(k\sqrt{(\xi-x)^2+(\eta-y)^2}\right)$$

(d) Write the equation for $u(\xi,\eta)$ in terms of boundary integrals (it will simplify quite a bit). Include the Green's function from the last step using the notation $Y_0' = \frac{\partial Y_0}{\partial y}$.

Solution

$$u(\xi,\eta)=\frac{k\eta}{2}\int_{-\infty}^{\infty}\frac{f(x)\,Y_0'\left(k\sqrt{(\xi-x)^2+\eta^2}\right)}{\sqrt{(\xi-x)^2+\eta^2}}dx$$

3. Finite Element Method

Consider the Helmholtz equation from Problem 1 on the two-dimensional domain Ω shown in Figure 1. The nodes are the numbers without boxes and the elements are the numbers in boxes. The flux $q = \frac{\partial u}{\partial n}$ is specified everywhere except on Γ_{59} and Γ_{45}. On these two boundaries $u = 0$.

(a) Derive the variational boundary value problem (the weak form, modified so that test and trial functions come from the samem space) to be used in FEM.

Solution Using the Gauss divergence theorem we obtain

$$\int_{\Omega}[\nabla v\cdot\nabla u-k^2vu]\,dx\,dy=\int_{\partial\Omega}v\frac{\partial u}{\partial n}\,ds.$$

(b) What assumptions do you need to make on test and trial functions u and v to be able to solve this problem? Please include ALL assumptions to define a suitable class or functions, as well as assumptions specific to this problem. Are the trial and test functions the same? Why or why not?

solution The test and trial functions must be square integrable, along with their first partial derivatives. Furthermore, the test and trial functions must be zero on the boundaries Γ_{59} and Γ_{45},

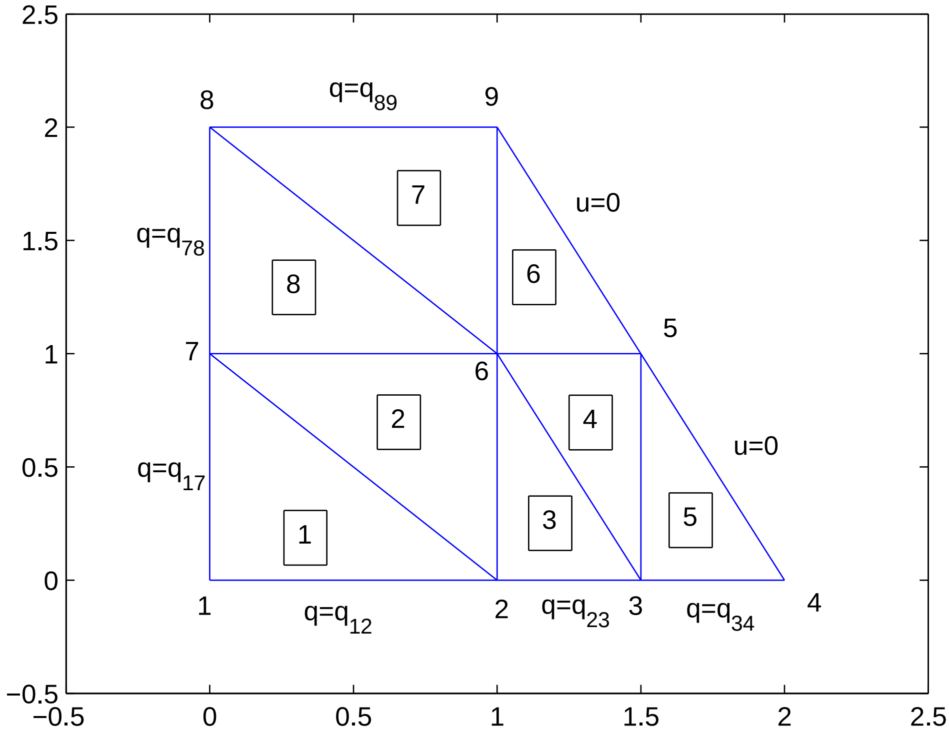

Figure 1: FEM domain Ω for Problem 3

so the test and trial functions are the same. *Note:* if the Dirichlet boundary conditions were not homogeneous, then only the test functions would be zero on these boundaries, and then the test and trial functions would not be exactly the same.

(c) Consider the 3×3 element matrix that results from linear functions on a generic element. Do you think the matrix will be symmetric? Why or why not?

Solution The matrix will be symmetric. This is a consequence of the fact that the Helmholtz operator is self-adjoint.

(d) Ignoring the boundary conditions for the moment, what will the stiffness matrix and load vector look like for linear functions? You don't have to compute anything, just build K and $\mathbf{f}$ with 'x' wherever a nonzero entry should be.

Solution For each element e, place an 'x' in the entry K_{ij} where i and j are the indices of any two of the nodes on the boundary of e. It follows that the structure of the matrix is

$$K = \begin{bmatrix} X & X & 0 & 0 & 0 & 0 & X & 0 & 0 \\ X & X & X & 0 & 0 & X & X & 0 & 0 \\ 0 & X & X & X & X & X & 0 & 0 & 0 \\ 0 & 0 & X & X & X & 0 & 0 & 0 & 0 \\ 0 & 0 & X & X & X & X & 0 & 0 & X \\ 0 & X & X & 0 & X & X & X & X & X \\ X & X & 0 & 0 & 0 & X & X & X & 0 \\ 0 & 0 & 0 & 0 & 0 & X & X & X & X \\ 0 & 0 & 0 & 0 & X & X & 0 & X & X \end{bmatrix}.$$

The load vector $\mathbf{f}$ will be zero, because there is no source term.

(e) Which entries of the stiffness matrix and the load vector will be changed when the boundary conditions are implemented? Mark the changes on your answer from the last part of this question. If it is necessary to remove any rows or columns from the matrix cross them out.

Solution No change needs to be made to the stiffness matrix due to the Neumann boundary conditions, but elements 1,2,3,7 and 8 of the load vector will have boundary terms added to them. Because of the Dirichlet conditions, rows and columns 4, 5 and 9 of K will be removed, along with the corresponding rows of $\mathbf{f}$. However, because they are homogeneous, no changes need to be made to the remaining entries of $\mathbf{f}$.

4. Wavelet Analysis

(a) Consider the piecewise constant signal

$$g(t) = \{1, 0.5, -0.75, 0.25\}.$$

The signal is shown in Figure 2. Assume that the signal is defined on the interval $[0, 1/2]$. This signal can be expressed as a sum of Haar smoothing functions at a particular scale M,

$$g(t) = \sum_{n=-\infty}^{\infty} \phi_{M,n}(t) s_{M,n}.$$

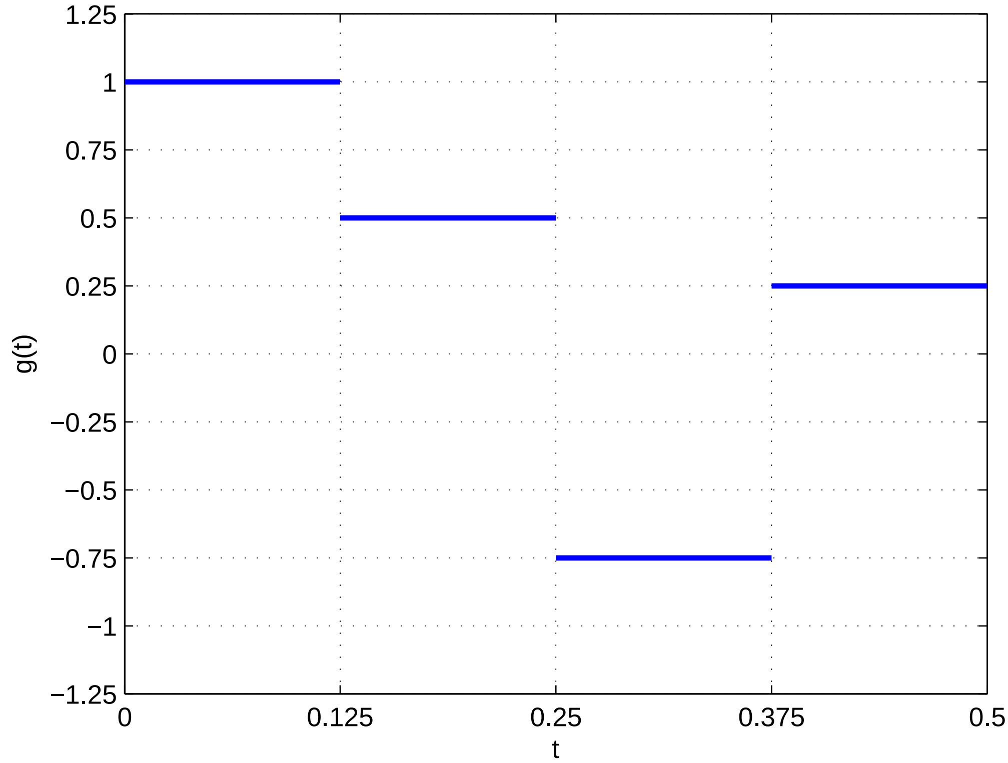

Figure 2: Signal for Problem 4

provided that this scale is sufficiently fine. What is the largest value of M for which this is possible? In other words, what is the largest M such that $g \in V_M$?

Solution The signal is piecewise constant on intervals of width $1/8 = 2^{-3}$, and V_M consists of functions that are nonzero constant on intervals of width 2^M, so $M = -3$.

(b) Perform the Haar decomposition of g to compute the detail coefficients $\{d_{M+1,n}\}$, $\{d_{M+2,n}\}$, and $\{s_{M+2,n}\}$, where M is the scale determined in part 4a.

Solution Because the signal is supported only on $[0, 1/2)$, and

$$d_{m,n} = 2^{-m/2}\left[\int_{n2^m}^{(n+1/2)2^m} g(t)\,dt - \int_{(n+1/2)2^m}^{(n+1)2^m} g(t)\,dt\right],$$

it follows that for $m = -2$, we only need to compute coefficients for $n = 0$ and $n = 1$. We have

$$d_{-2,0} = 2\left[\int_0^{1/8} g(t) - \int_{1/8}^{1/4} g(t)\,dt\right] = 2\frac{1}{8}(1 - 0.5) = \frac{1}{8},$$

$$d_{-2,1} = 2\left[\int_{1/4}^{3/8} g(t) - \int_{3/8}^{1/2} g(t)\,dt\right] = 2\frac{1}{8}(-0.75 - 0.25) = -\frac{1}{4}.$$

Using the formula

$$s_{m,n} = 2^{-m/2}\int_{n2^m}^{(n+1)2^m} g(t)\,dt,$$

we also obtain

$$s_{-2,0} = 2\int_0^{1/4} g(t)\,dt = 2\frac{1}{8}(1 + 0.5) = \frac{3}{8},$$

$$s_{-2,1} = 2\int_{1/4}^{1/2} g(t)\,dt = 2\frac{1}{8}(-0.75 + 0.25) = -\frac{1}{8},$$

which yields a smoothed signal belonging to V_{-2},

$$g_{-2}(t) = s_{-2,0}\phi_{-2,0}(t) + s_{-2,1}\phi_{-2,1}(t) = \begin{cases} 3/4 & 0 \le x < 1/4 \\ -1/4 & 1/4 \le x < 1/2 \end{cases}.$$

We can work with this simpler signal to complete the decomposition. We have

$$d_{-1,0} = \sqrt{2}\int_0^{1/4} g_{-2}(t)\,dt - \int_{1/4}^{1/2} g_{-2}(t)\,dt = \sqrt{2}\frac{1}{4}(3/4 - (-1/4)) = \frac{\sqrt{2}}{4},$$

$$s_{-1,0} = \sqrt{2}\int_0^{1/2} g_{-2}(t)\,dt = \sqrt{2}\frac{1}{4}(3/4 - 1/4) = \frac{\sqrt{2}}{8}.$$

(c) Compress the signal by setting to zero the detail coefficients which satisfy

$$|d_{m,n}| \le 0.25.$$

Reconstruct the compressed signal $\tilde{g}$ using the relation

$$\sum_{n=-\infty}^{\infty} s_{m,n}\phi_{m,n}(t) = \sum_{n=-\infty}^{\infty} s_{m+1,n}\phi_{m+1,n}(t) + \sum_{n=-\infty}^{\infty} d_{m+1,n}\psi_{m+1,n}(t).$$

Solution The coefficients $d_{-2,0}$ and $d_{-2,1}$ are dropped, while $d_{-1,0}$ is retained. Because $g \in V_{-3}$, $\tilde{g} \in V_{-3}$. Let $\tilde{g}$ have approximation coefficients $\tilde{s}_{m,n}$ and detail coefficients $\tilde{d}_{m,n}$. It follows from the fact that $\tilde{d}_{-2,n} = 0$ for all n that

$$\begin{aligned}
\tilde{g}(t) &= \sum_{n=-\infty}^{\infty} \tilde{s}_{-3,n}\phi_{-3,n}(t) \\
&= \sum_{n=-\infty}^{\infty} \tilde{s}_{-2,n}\phi_{-2,n}(t) + \sum_{n=-\infty}^{\infty} \tilde{d}_{-2,n}\psi_{-2,n}(t) \\
&= \sum_{n=-\infty}^{\infty} \tilde{s}_{-1,n}\phi_{-1,n}(t) + \sum_{n=-\infty}^{\infty} \tilde{d}_{-1,n}\psi_{-1,n}(t) \\
&= s_{-1,0}\phi_{-1,0}(t) + d_{-1,0}\phi_{-1,0}(t) \\
&= \frac{\sqrt{2}}{8}\sqrt{2}\phi(2t) + \frac{\sqrt{2}}{4}\sqrt{2}\phi(2t) \\
&= \frac{1}{4}\phi(2t) + \frac{1}{2}\psi(2t) \\
&= \begin{cases} 3/4 & 0 \le t < 1/4 \\ -1/4 & 1/4 \le t < 1/2 \end{cases} .
\end{aligned}$$

(d) If $\tilde{g}$ is a compression of a signal g, then the compression ratio is given by

$$100\frac{\|\tilde{g}\|_{L_2}^2}{\|g\|_{L_2}^2} = 100\frac{\sum_{m,n=-\infty}^{\infty} \tilde{d}_{m,n}^2}{\sum_{m,n=-\infty}^{\infty} d_{m,n}^2},$$

where $\{\tilde{d}_{m,n}\}$ is the set of detail coefficients for the compressed signal. Compute and interpret the compression ratio for the compression performed in part 4c. *Hint:* $\tilde{d}_{M+3,0} = d_{M+3,0} = \sqrt{2}s_{M+2,0}$.

Solution We have $\tilde{d}_{0,0} = d_{0,0} = \sqrt{2}s_{-1,0} = 1/4$. Therefore the compression ratio is

$$100\frac{(1/4)^2 + (\sqrt{2}/4)^2}{(1/4)^2 + (\sqrt{2}/4)^2 + (1/8)^2 + (-1/4)^2} = 70.59.$$

This is a low compression ratio, indicating that a higher threshold should have been chosen.