Ballistics

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Outline

Dynamics

Simulations

Targeting

Robust targeting

Position, velocity, and force

- a projectile moves in 2-dimensional space
  (for simplicity; real ones move in 3-dimensional space)
- sample position and velocity at times $\tau = 0, h, 2h, \ldots$
- 2-vector p_t is position at time $\tau = th$ for $t = 0, 1, \ldots$
- 2-vector v_t is velocity at time $\tau = th$ for $t = 0, 1, \ldots$
- 2-vector f_t is total force acting on projectile at time $\tau = th$
- 4-vector $x_t = \begin{bmatrix} p_t \\ v_t \end{bmatrix}$ is projectile *state* at time $\tau = th$

Force model

$$f_t = mg - \eta(v_t - w)$$

- 2-vector $g = (0, -9.8)$ is gravity
- 2-vector w is wind velocity (assumed constant)
- $v_t - w$ is relative velocity of projectile through air
- $\eta \in \mathbf{R}$ is *drag coefficient*
- $\eta(v_t - w)$ is *drag force*
- m is projectile mass
- 'ballistic' means the projectile has no other force acting on it (*e.g.*, thrust or propulsion)

Dynamics

- approximating velocity as constant over time interval $th \leq \tau \leq (t+1)h$,
$$p_{t+1} = p_t + hv_t$$
- approximating force as constant over the time interal,
$$\begin{aligned} v_{t+1} &= v_t + (h/m)f_t \\ &= (1 - h\eta/m)v_t + (hg + h\eta w/m) \end{aligned}$$
- more compactly: $x_{t+1} = Ax_t + b$, with
$$A = \begin{bmatrix} 1 & 0 & h & 0 \\ 0 & 1 & 0 & h \\ 0 & 0 & 1 - h\eta/m & 0 \\ 0 & 0 & 0 & 1 - h\eta/m \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \\ hg_1 + h\eta w_1/m \\ hg_2 + h\eta w_2/m \end{bmatrix}$$

Propagating state through time

- to propagate forward T time steps

$$
\begin{aligned}
x_1 &= Ax_0 + b \\
x_2 &= A(Ax_0 + b) + b = A^2x_0 + Ab + b \\
&\vdots \\
x_T &= A^T x_0 + (A^{T-1} + \cdots + A + I)b
\end{aligned}
$$

Outline

Simulation parameters

- let's look at some trajectories, with parameters

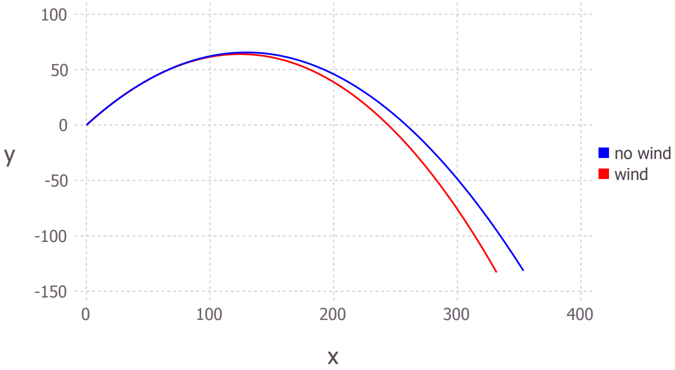
$$m = 5, \quad T = 100, \quad h = 0.1, \quad \eta = 0.05, \quad p_0 = 0$$

- we'll use various values of initial velocity v_0, expressed in terms of
  - initial speed $\|v_0\|$
  - elevation $\theta = \tan^{-1}((v_0)_2/(v_0)_1)$
- we'll vary the wind velocity w too

Simulation: with and without wind

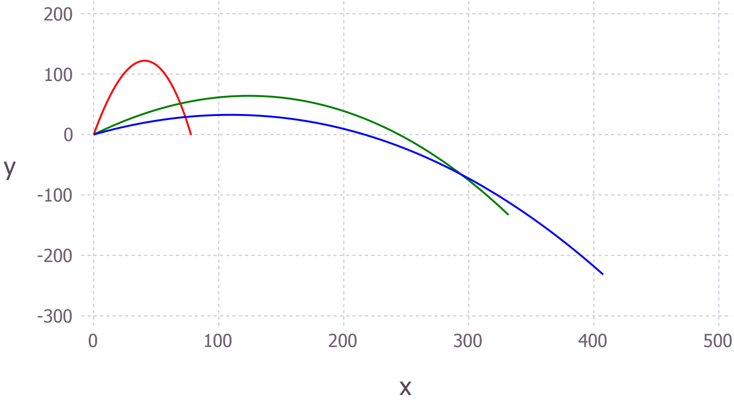
- initial speed $\|v_0\| = 50$, elevation $\theta = 45°$
- no wind: $w = (0, 0)$
- with wind: $w = (-10, 0)$

(all future simulations include wind)



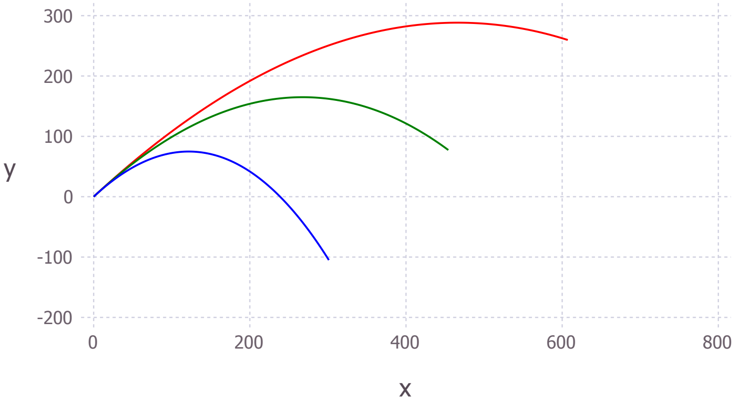
Simulation: varying elevation

- $\|v_0\| = 50$
- $\theta = 30°,\ 45°,\ 80°$



Simulation: varying speed

- $\theta = 50$
- $\|v_0\| = 50,\ 75,\ 100$



Outline

Targeting problem

- given
  - initial position p_0
  - parameters h, m, w, η
  - flight time Th
  - desired final position (‘target’) p_T
- find initial velocity v_0

- please note
  - this is not used for socially positive purposes
  - but it is one of the first historical applications

Final state

- final state is

$$
\begin{aligned}
x_T &= A^T x_0 + (A^{T-1} + \cdots + A + I)b \\
&= F x_0 + j
\end{aligned}
$$

where

$$F = A^T, \qquad j = (A^{T-1} + \cdots + A + I)b$$

  - 4×4 matrix F maps initial state to final state
  - 4-vector j is effect of gravity, wind on final state

Final position

- final position is

$$p_T = F_{11}p_0 + F_{12}v_0 + j_1$$

(F_{11} and F_{12} are 2×2 subblocks of F)

- write as $p_T = Cv_0 + d$, where $C = F_{12}$, $d = F_{11}p_0 + j_1$
- solving for v_0 we have (assuming $C = F_{12}$ is invertible)

$$v_0 = C^{-1}(p_T - d)$$

(note that C and d are known)

- gives formula for choosing v_0 (hence, $\|v_0\|$ and θ)

Outline

Robust ballistics

- suppose we have uncertainty in the wind, drag coefficient, ...
- uncertainty is modeled as K *scenarios* (particular values of parameters)
  - each scenario has its own $A^{(j)}$, $b^{(j)}$
  - hence its own $C^{(j)}$, $d^{(j)}$
- *robust targetting*: choose a single v_0 to minimize mean-square targetting error

$$\frac{1}{K}\sum_{j=1}^{K}\|C^{(j)}v_0 + d^{(j)} - p_T\|^2$$

Sample simulations

various masses, drag coefficients, and wind, with $T = 100$