Images

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Outline

Monochrome images

- a.k.a. *monochrome* or *gray-scale* image
- image represented by its brightness values at an array of $m \times n$ locations (pixels)
- typical sizes
  - thumbnail: 16×16, 64×64, 128×128
  - 4K $\times$ 6K $=$ 24M pixels
  - HD is 1280×720 pixels
- can represent by an $m \times n$ matrix X or by a single vector $x \in \mathbf{R}^{mn}$ with some encoding of the pixel locations, *e.g.*,

$$X_{ij} = x_k, \quad k = m(j-1) + i, \quad k = 1, \ldots, mn$$

(this stacks the columns of X, from left to right)

Brightness values

- x_i is brightness of pixel i
- typically $0 \leq x_i \leq 1$ where 0 is black and 1 is white
- values outside $[0, 1]$ are clipped (so $x_i < 0$ shows up as black)
- if x is an image, $-x$ is completely black
- *negative image* is given by $\mathbf{1} - x$
- $\mathbf{avg}(x)$ is average intensity (brightness) of image
- $\mathbf{std}(x)$ corresponds to image *constrast*

Scaling, shifting, and adding images

- what does image

$$y = a(x - \mathbf{avg}(x)\mathbf{1}) + (\mathbf{avg}(x) + b)\mathbf{1} = ax + c\mathbf{1}$$

  ($c = (1-a)\,\mathbf{avg}(x) + b$) look like?
  - a scale contrast
  - b shifts brightness

- $y_i = x_i^\gamma$ is called γ-*correction* (widely used)

- if x and y are images, $x + y$ is perceived as composite or combination of the images (and isn't natural, except in some cases)

Examples

original

original + 0.5

(original – 0.4) * 10

Examples

pumpkins

flowers

(pumpkins+flowers)/2

Color images

- humans perceive 3 colors, which can be represented in different ways (*e.g.*, RGB, CMYK)
- most common is RGB (Red-Green-Blue)
- color represented as a 3-vector (r, g, b), with r, g, b between 0 and 1
  - $(1, 0, 0)$ is bright red
  - $(1, 0, 1)$ is bright purple
  - $(0.2, 0.2, 0.2)$ is a gray
- $m \times n$ image given by 3 $m \times n$ matrices or one vector $x \in \mathbf{R}^{3mn}$

Colors

- (1,0,0), (0,1,0), (0,0,1)
- (1,1,0), (0,1,1), (1,0,1)
- (0.2,0.2,0.2), (0.5,0.5,0.5), (0.75,0.75,0.75)

Color images

Converting color to monochrome

- color pixel values converted to monochrome using $y_i = w^T(r_i, g_i, b_i)$
  - obvious choice: $w = (1/3, 1/3, 1/3)$
  - another common choice: $w = (0.299, 0.587, 0.114)$
  - other choices used for special effects

Converting color to monochrome

Video

- video is represented as a sequence of images captured periodically
- each image is called a *frame*
- typical frame rates: 24, 30, or 60 frames per second

Outline

Representation

Linear operations

In-painting

Image de-blurring

Linear image mappings

- for y and x images, linear mapping $y = Ax$ can represent many common operations on images
  - color to monochrome conversion
  - color correction
  - any mapping from original to distorted pixel locations (*e.g.*, flipping, stretching)
  - blurring
  - changing to lower or higher resolution
  - vertical and horizontal differencing

Moving pixels

- pixel at location i in image y is the pixel at value $j = d(i)$ in image x
- $d(i)$ gives *distortion map*
- examples: flipping, zooming, rotating, shifting, key correction
- some issues/details:
  - we'll need to approximate the location of the pixels
  - we need to do something with y pixels that don't correspond to any x pixels
- $y = Ax$, where ith row of A is $e_{d(i)}^T$
  (or 0, if y_i doesn't correspond to any x pixel)

Flipping images

original image

horizontal flip

vertical flip

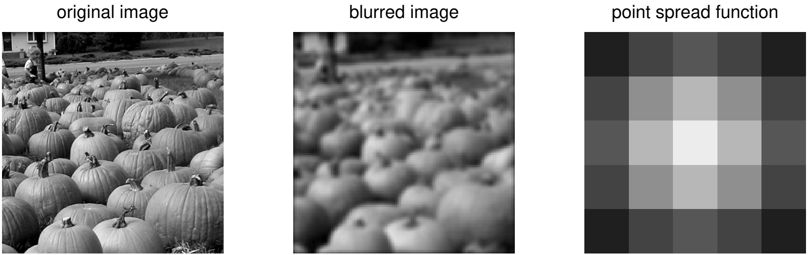
Blurring images

- represent image as $m \times n$ matrix X
- represent blur *point spread function* as $p \times q$ matrix B
- blurred image is given by Y with

$$Y_{ij} = \sum_{k,l} X_{i-k+1,j-l+1} B_{k,l}$$

  where
  - the sum is over all integers k, l
  - we interpret X_{ij} and $B_{k,l}$ as zero when the indices are out of range
- called *2-D convolution* of X and B, denoted $Y = X * B$ or $Y = A \star B$
- blurring is model of effects of optical imperfections, motion blur, ...

Blurring images



Horizontal and vertical differences

- X is $m \times n$ image (matrix), x its mn-vector representation
- horizontal first order difference is $m \times (n-1)$ matrix Y with

$$Y_{ij} = X_{i,j+1} - X_{i,j}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n-1$$

- vertical first order difference is $(m-1) \times n$ matrix Z with

$$Z_{ij} = X_{i+1,j} - X_{i,j}, \quad i = 1, \ldots, m-1, \quad j = 1, \ldots, n$$

- these are linear operations, so we have

$$y = D^{\text{horiz}} x, \qquad z = D^{\text{vert}} x$$

  for an $m(n-1)$-matrix D^{horiz} and an $(m-1)n$-matrix D^{vert}
- each row contains one $+1$ and one -1

Horizontal and vertical differences

(shown for 3×3 image)

$$
\begin{aligned}
D^{\text{horiz}} &= \begin{bmatrix}
-1 & & & +1 & & & & & \\
 & -1 & & & +1 & & & & \\
 & & -1 & & & +1 & & & \\
 & & & -1 & & & +1 & & \\
 & & & & -1 & & & +1 & \\
 & & & & & -1 & & & +1
\end{bmatrix} \\
D^{\text{vert}} &= \begin{bmatrix}
-1 & +1 & & & & & & & \\
 & -1 & +1 & & & & & & \\
 & & & -1 & +1 & & & & \\
 & & & & -1 & +1 & & & \\
 & & & & & & -1 & +1 & \\
 & & & & & & & -1 & +1
\end{bmatrix}
\end{aligned}
$$

Horizontal and vertical differences

Dirichlet energy

- the *DIrichlet energy* (also called *Laplacian*) is

$$\begin{aligned}\mathcal{D}(x) &= \|D^{\text{horiz}}x\|^2 + \|D^{\text{vert}}x\|^2 \\ &= \sum_{i=1}^{m-1}\sum_{j=1}^{n-1}\left((X_{i+1,j}-X_{i,j})^2 + (X_{i,j+1}-X_{i,j})^2\right)\end{aligned}$$

  (we also write $\mathcal{D}(X)$)
- $\mathcal{D}(X)$ is a measure of roughness of the image X
  - $\mathcal{D}(X)$ is small when the image is smooth
  - $\mathcal{D}(X) = 0$ only if the image is constant
- $\mathcal{D}(X)$ is used as a *regularizer*

Outline

Representation

Linear operations

In-painting

Image de-blurring

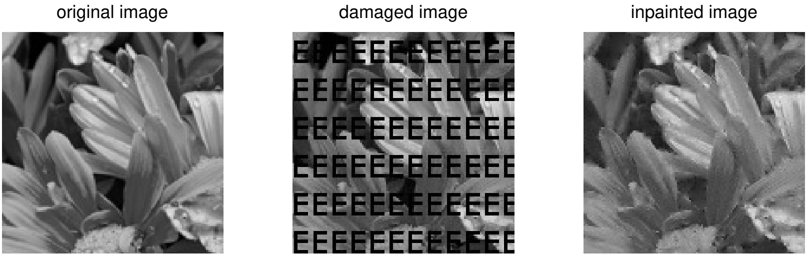
In-painting

- we are given an image with some pixels values unknown
- *in-painting* means to guess values of the unknown pixels so the recovered image looks good or natural
- in example below, unknown values are shown as black

Least-squares in-painting

- corrupted/damaged image is given by $m \times n$ matrix X^{corr}
- $\mathcal{K} \subset \{1, \ldots, m\} \times \{1, \ldots, n\}$ are the indices of known pixels
- we need to choose an image X that agrees with the given image on known pixels: $X_{ij} = X_{ij}^{\mathrm{corr}}$, $(i, j) \in \mathcal{K}$
- we'll choose X to minimize $\mathcal{D}(X)$, the sum square deviation of all pixel values from their neighbors (small $\mathcal{D}(X)$ gives a smooth image)
- a least-squares problem (variables are unknown pixel values)

In-painting



Outline

Corrupted image

- y is a linear function of x^{true}, with noise:

$$y = Ax^{\text{true}} + v$$

- y is the *corrupted* image, which we have
- x^{true} is the original image, which we want to guess/recover
- v is a noise, which we assume is small
- A is a (known) matrix, often a blurring operator
- *image de-blurring* is guessing x^{true}
- even if A is invertible, the guess $x = A^{-1}y$ could look very bad

Least-squares de-blurring

- *least-squares de-blurring*: choose x to minimize

$$\|Ax - y\|^2 + \lambda \mathcal{D}(x)$$

- first term is $\|v\|^2$
- $\lambda > 0$ is a regularization parameter
  - large λ makes x smooth
  - small λ makes $\|Ax - y\|^2$ small

De-blurring example

original image

corrupted image

deblurred image with lambda = 0.03

De-blurring: Effect of regulariziton

lambda = 0.0003

lambda = 0.03

lambda = 3