Portfolio Optimization

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Outline

Return of an asset over one period

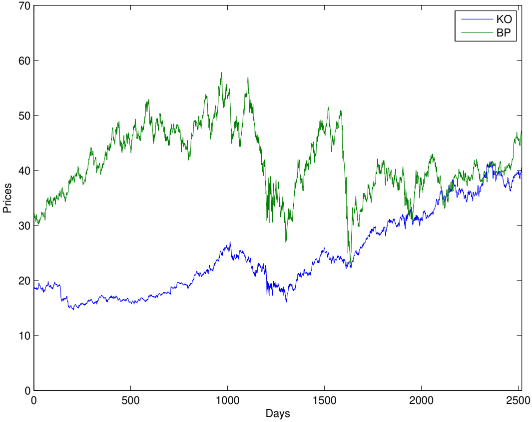
- *asset* can be stock, bond, real estate, commodity, ...
- invest in a single asset over period (quarter, week, day, ...)
- buy q shares at price p (at beginning of investment period)
- $h = pq$ is dollar value of holdings
- sell q shares at new price p^+ (at end of period)
- profit is $qp^+ - qp = q(p^+ - p) = \frac{p^+ - p}{p} h$
- define **return** $r = \frac{p^+ - p}{p} = \dfrac{\text{profit}}{\text{investment}}$
- profit $= rh$
- example: invest $h = \$1000$ over period, $r = +0.03$: profit $= \$30$

Short positions

- basic idea: holdings h and share quantities q are **negative**
- called *shorting* or *taking a short position on* the asset
  (h or q positive is called a *long position*)
- how it works:
  - you borrow q shares at the beginning of the period and sell them at price p
  - at the end of the period, you have to buy q shares at price p^+ to return them to the lender
- all formulas still hold, *e.g.*, profit $= rh$
- example: invest $h = -\$1000$, $r = -0.05$: profit $= +\$50$
- no limit to how much you can lose when you short assets
- normal people (and mutual funds) don't do this; hedge funds do

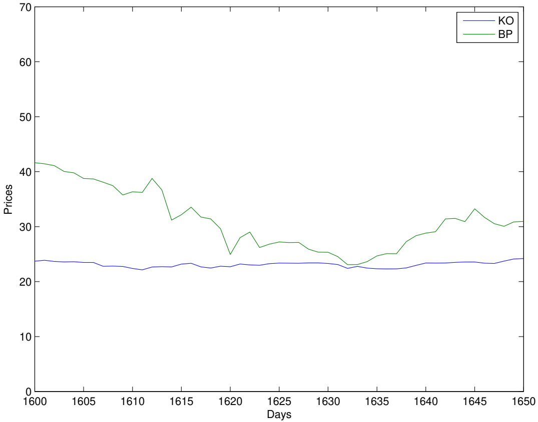
Examples

prices of BP (BP) and Coca-Cola (KO) for last 10 years



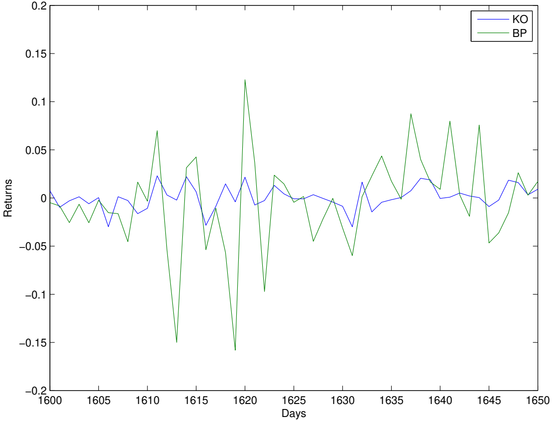
Examples

zoomed in to 10 weeks



Examples

returns over the same period



Return and risk

- suppose r is time series (vector) of returns
- **average return** or just **return** is $\mathbf{avg}(r)$
- **risk** is $\mathbf{std}(r)$
- these are the per-period return and risk

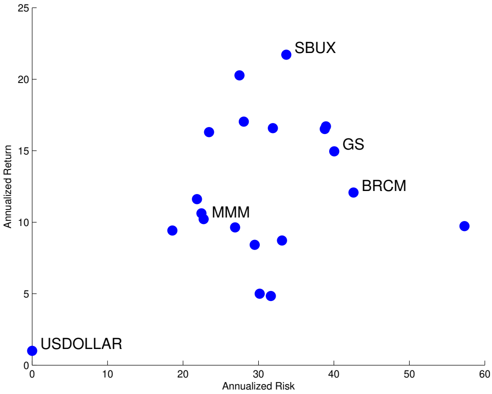
Annualized return and risk

- mean return and risk are often expressed in **annualized form** (*i.e.*, per year)
- if there are P trading periods per year
  - annualized return $= P\,\mathbf{avg}(r)$
  - annualized risk $= \sqrt{P}\,\mathbf{std}(r)$

  (the squareroot in risk annualization comes from the assumption that the fluctuations in return around the mean are independent)
- if returns are daily, with 250 trading days in a year
  - annualized return $= 250\,\mathbf{avg}(r)$
  - annualized risk $= \sqrt{250}\,\mathbf{std}(r)$

Risk-return plot

- annualized risk versus annualized return of various assets
- up (high return) and left (low risk) is good



Outline

Return and risk

Portfolio investment

Portfolio optimization

Portfolio of assets

- n assets
- n-vector h_t is dollar value holdings of the assets
- total portfolio value: $V_t = \mathbf{1}^T h_t$ (we assume positive)
- $w_t = (1/\mathbf{1}^T h_t)h_t$ gives **portfolio weights** or **allocation** (fraction of total portfolio value)
- $\mathbf{1}^T w_t = 1$

Examples

- $(h_3)_5 = -1000$ means you short asset 5 in investment period 3 by \$1,000
- $(w_2)_4 = 0.20$ means 20% of total portfolio value in period 2 is invested in asset 4
- $w_t = (1/n, \ldots, 1/n)$, $t = 1, \ldots, T$ means total portfolio value is equally allocated across assets in all investment periods

Portfolio return and risk

- asset returns in period t given by n-vector $\tilde{r}_t$
- dollar profit (increase in value) over period t is $\tilde{r}_t^T h_t = V_t \tilde{r}_t^T w_t$
- portfolio return (fractional increase) over period t is

$$\frac{V_{t+1} - V_t}{V_t} = \frac{V_t(1 + \tilde{r}_t^T w_t) - V_t}{V_t} = \tilde{r}_t^T w_t$$

- $r_t = \tilde{r}_t^T w_t$ is called **portfolio return** in period t
- r is T-vector of portfolio returns
- $\mathbf{avg}(r)$ is portfolio return (over periods $t = 1, \ldots, T$)
- $\mathbf{std}(r)$ is portfolio risk (over periods $t = 1, \ldots, T$)

Compounding and re-investment

- $V_{T+1} = V_1(1+r_1)(1+r_2)\cdots(1+r_T)$
- product here is called **compounding**
- for $|r_t|$ small (say, ≤ 0.01) and T not too big,

$$V_{T+1} \approx V_1(1 + r_1 + \cdots + r_T) = V_1(1 + T\,\mathbf{avg}(r))$$

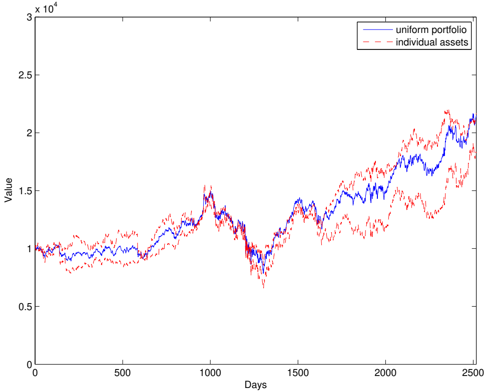
- so high average return corresponds to high final portfolio value
- $V_t \leq 0$ (or some small value like $0.1V_1$) called **going bust** or **ruin**

Constant weight portfolio

- constant weight vector w, *i.e.*, $w_t = w$ for $t = 1, \ldots, T$
- requires **rebalancing** to weight w after each period
- define $T \times n$ asset returns matrix R with rows $\tilde{r}_t^T$
- so R_{tj} is return of asset j in period t
- then $r = Rw$

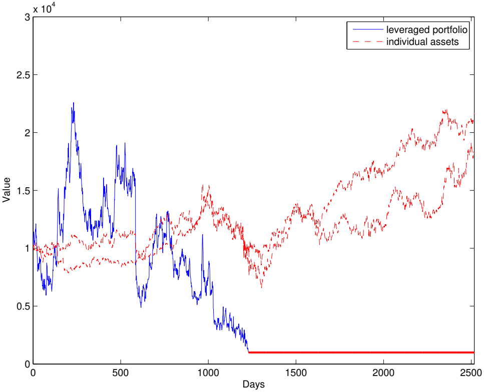
Cumulative value plot

- assets are Coca-Cola (KO) and Microsoft (MSFT)
- constant weight portfolio with $w = (0.5, 0.5)$
- $V_1 = \$10000$ (by tradition)



Cumulative value plot

- $w = (-3, 4)$
- portfolio **goes bust** (drops to 10% of starting value)



Outline

Portfolio optimization

- how should we choose the portfolio weight vector w?
- we want high (mean) portfolio return, low portfolio risk

- we know past **realized asset returns** but not future ones
- we will choose w that would have worked well on past returns
- ...and hope it will work well going forward (just like data fitting)

Portfolio optimization

$$\begin{array}{ll} \text{minimize} & \mathbf{std}(Rw)^2 = (1/T)\|Rw - \rho\mathbf{1}\|^2 \\ \text{subject to} & \mathbf{1}^T w = 1, \quad \mathbf{avg}(Rw) = \rho \end{array}$$

- w is the weight vector we seek
- R is the returns matrix for **past returns**
- Rw is the (past) portfolio return time series
- require mean (past) return ρ
- we minimize risk for specified value of return

- we are really asking what **would have been** the best constant allocation, had we known future returns

Portfolio optimization via least squares

$$\begin{array}{ll} \text{minimize} & \|Rw - \rho\mathbf{1}\|^2 \\ \text{subject to} & \left[\begin{array}{c} \mathbf{1}^T \\ \mu^T \end{array}\right] w = \left[\begin{array}{c} 1 \\ \rho \end{array}\right] \end{array}$$

- $\mu = R^T\mathbf{1}/T$ is n-vector of (past) asset returns
- ρ is required (past) portfolio return
- equality constrained least squares problem, with solution

$$\left[\begin{array}{c} w \\ z_1 \\ z_2 \end{array}\right] = \left[\begin{array}{ccc} 2R^TR & \mathbf{1} & \mu \\ \mathbf{1}^T & 0 & 0 \\ \mu^T & 0 & 0 \end{array}\right]^{-1} \left[\begin{array}{c} 2\rho T\mu \\ 1 \\ \rho \end{array}\right]$$

Examples

- optimal w for annual return 1% (last asset is risk-less with 1% return)

$$w = (0.0000, 0.0000, 0.0000, \ldots, 0.0000, 0.0000, 1.0000)$$

- optimal w for annual return 13%

$$w = (0.0250, -0.0715, -0.0454, \ldots, -0.0351, 0.0633, 0.5595)$$

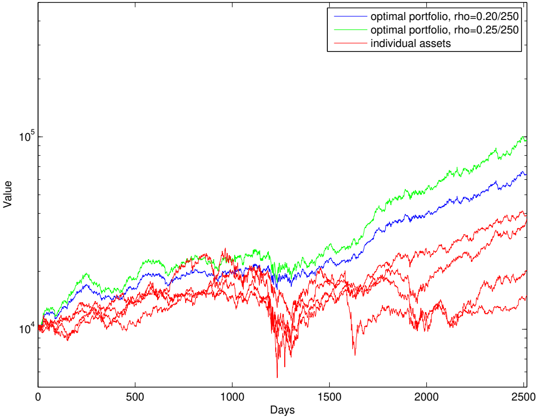
- optimal w for annual return 25%

$$w = (0.0500, -0.1430, -0.0907, \ldots, -0.0703, 0.1265, 0.1191)$$

- asking for higher annual return yields
  - more invested in risky, but high return assets
  - larger short positions ('leveraging')

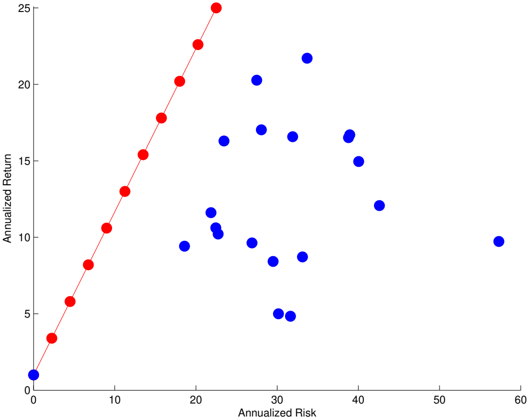
Cumulative value plots for optimal portfolios

cumulative value plot for optimal portfolios and some individual assets



Optimal risk-return curve

red curve obtained by solving problem for various values of ρ



Optimal portfolios

- perform significantly better than individual assets
- risk-return curve forms a straight line
  - one end of the line is the risk-free asset
- *two-fund theorem:* optimal portfolio w is an affine function in ρ

$$\begin{bmatrix} w \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2R^T R & \mathbf{1} & \mu \\ \mathbf{1}^T & 0 & 0 \\ \mu^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} R^T \mathbf{1} \\ 1 \\ \rho T \end{bmatrix}$$

The big assumption

- now we make the big assumption (BA):

  future returns will look something like past ones

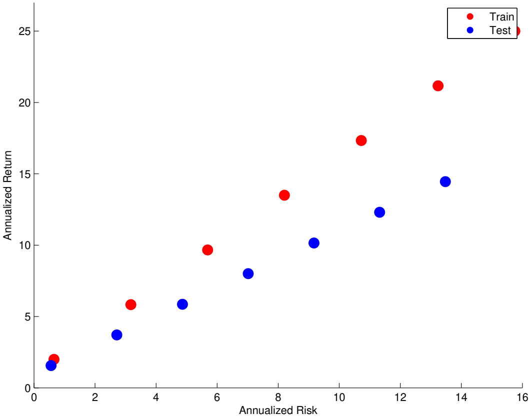
  - you are warned this is false, every time you invest
  - it is often reasonably true
  - in periods of 'market shift' it's much less true

- if BA holds (even approximately), then a good weight vector for past (realized) returns should be good for future (unknown) returns

- for example:
  - choose w based on last 2 years of returns
  - then use w for next 6 months

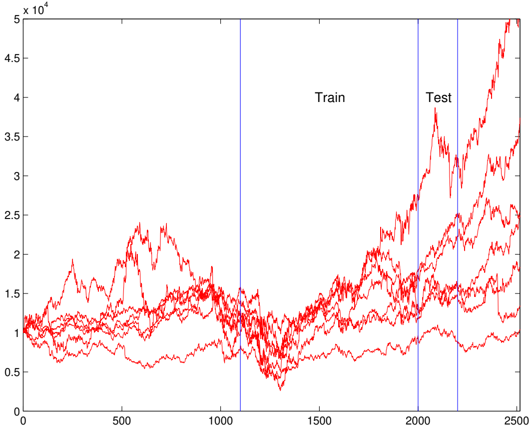
Optimal risk-return curve

- trained on 900 days (red), tested on the next 200 days (blue)
- here BA held reasonably well



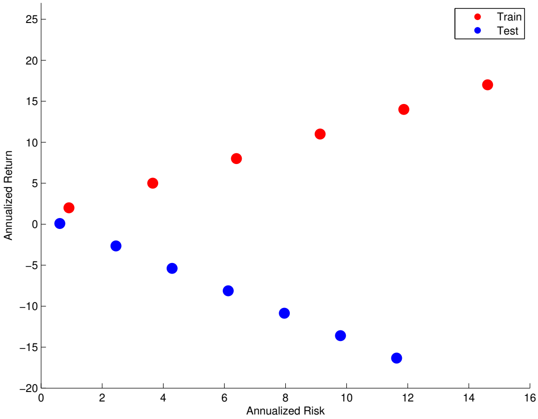
Optimal risk-return curve

- corresponding train and test periods



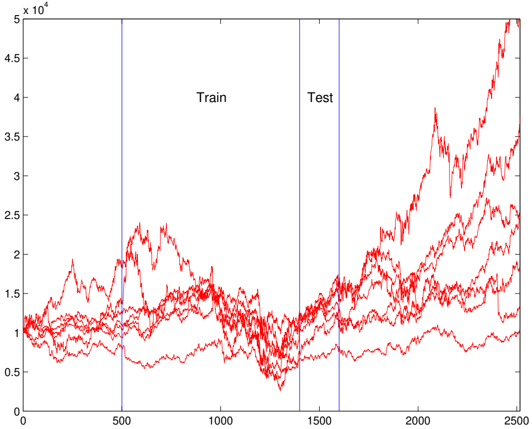
Optimal risk-return curve

- and here BA didn't hold so well
- (can you guess when this was?)



Optimal risk-return curve

- corresponding train and test periods



Rolling portfolio optimization

for each period t, find weight w_t using L past returns

$$r_{t-1}, \ldots, r_{t-L}$$

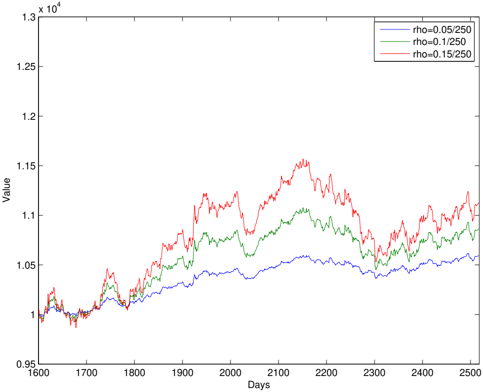
variations:

- update w every K periods (say, monthly or quarterly)
- add cost term $\kappa \|w_t - w_{t-1}\|^2$ to objective to discourage turnover, reduce transaction cost
- add logic to detect when the future is likely to not look like the past
- add 'signals' that predict future returns of assets

(…and pretty soon you have a quantitative hedge fund)

Rolling portfolio optimization example

- cumulative value plot for different target returns
- update w daily, using $L = 400$ past returns



Rolling portfolio optimization example

- same as previous example, but update w every quarter (60 periods)