Tomography

Keegan Go   Ahmed Bou-Rabee   Stephen Boyd

ENGR108
Stanford University

September 7, 2025

Tomography

- goal is to reconstruct or estimate a function $d : \mathbf{R}^2 \to \mathbf{R}$ from (possibly noisy) line integral measurements
- d is often (but not always) some kind of density
- we'll focus on 2-D case, but it can be extended to 3-D
- used in medicine, manufacturing, networking, geology
- best known application: CAT (computer-aided tomography) scan

Outline

Line integral measurements

Least squares reconstruction

Example

Line integral

- parameterize line ℓ in 2-D as

$$p(t) = (x_0, y_0) + t(\cos\theta, \sin\theta), \quad t \in \mathbf{R}$$

  - (x_0, y_0) is (any) point on the line
  - θ is angle of line (measured from horizontal)
  - parameter t is length along line

- line integral (of d, on ℓ) is

$$\int_\ell d = \int_{-\infty}^{\infty} d(p(t))\, dt$$

Line integral measurements

- we have m line integral measurements of d with lines $\ell_1, \ldots, \ell_m$
- ith measurement is

$$y_i = \int_{-\infty}^{\infty} d(p_i(t))\, dt + v_i, \quad i = 1, \ldots, m$$

  - $p_i(t)$ is parametrization of ℓ_i
  - v_i is the *noise* or *measurement error* (assumed to be small)
- vector of line integral measurements $y = (y_1, \ldots, y_m)$

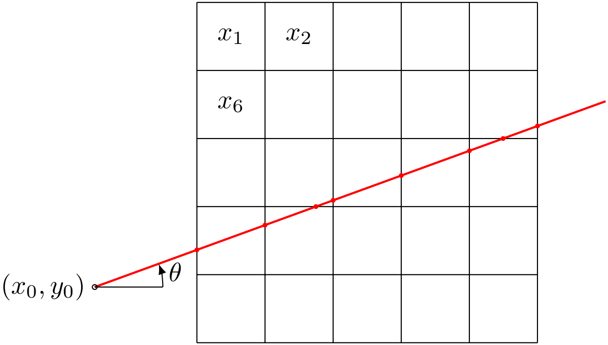
Discretization of d

- we assume d is constant on n pixels, numbered 1 to n
- represent (discretized) density function d by n-vector x
- x_i is value of d in pixel i
- line integral measurement y_i has form

$$y_i = \sum_{j=1}^{n} A_{ij} x_j + v_i$$

- A_{ij} is length of line ℓ_i in pixel j
- in matrix-vector form, we have $y = Ax + v$

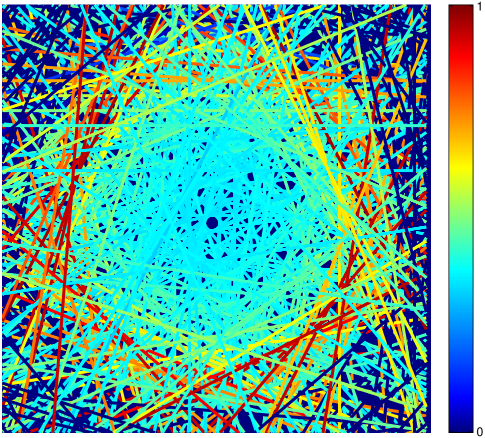
Illustration



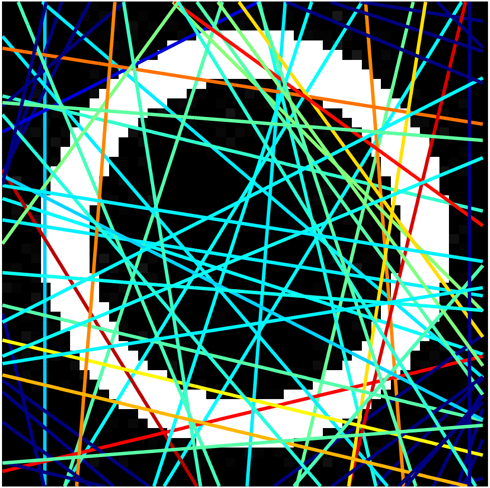
$$y = 1.06x_{16} + 0.80x_{17} + 0.27x_{12} + 1.06x_{13} + 1.06x_{14} + 0.53x_{15} + 0.54x_{10} + v$$

Example

image is 50×50
600 measurements shown

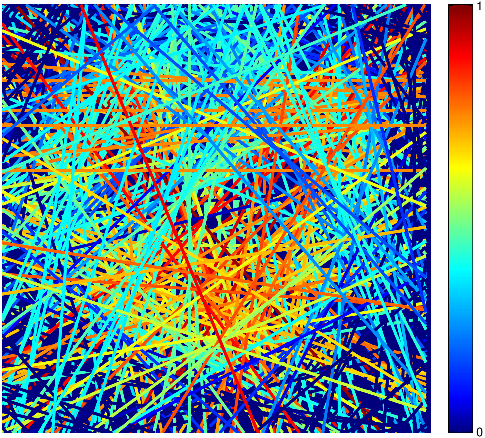


Example

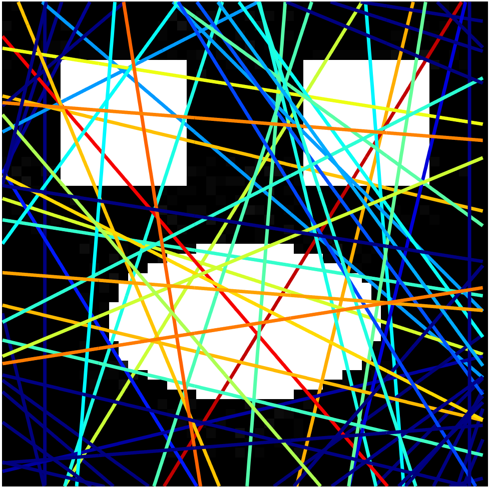


Another example

image is 50×50
600 measurements shown



Another example



Outline

Line integral measurements

Least squares reconstruction

Example

Smoothness prior

- we assume that image is not too rough, as measured by Dirichlet energy

$$\mathcal{D}(x) = \|D_\mathrm{v} x\|^2 + \|D_\mathrm{h} x\|^2$$

  - $D_h x$ gives first order difference in horizontal direction
  - $D_v x$ gives first order difference in vertical direction

- roughness measure is sum of squares of first order differences

- it is zero only when x is constant

Least squares reconstruction

- choose $\hat{x}$ to minimize

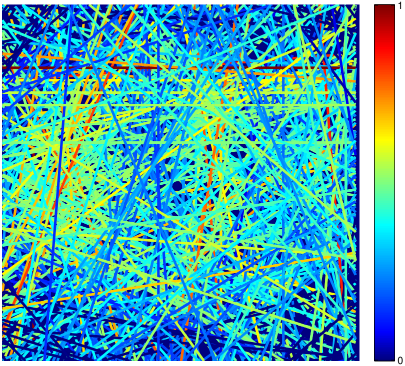
$$\|Ax - y\|^2 + \lambda \mathcal{D}(x)$$

  - first term is $\|v\|^2$, or deviation between what we observed (y) and what we would have observed without noise (Ax)
  - second term is roughness measure
- regularization parameter $\lambda > 0$ trades off measurement fit versus roughness of recovered image

Outline

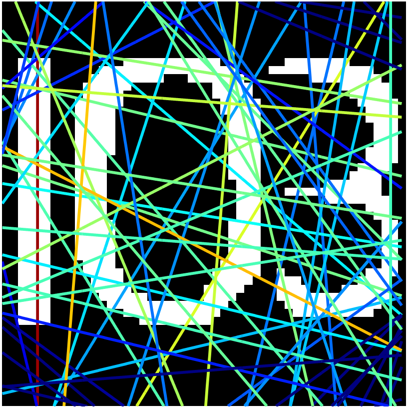
Example

- 50×50 pixels ($n = 2500$)
- 40 angles, 40 offsets ($m = 1600$ lines)
- 600 lines shown
- small measurement noise



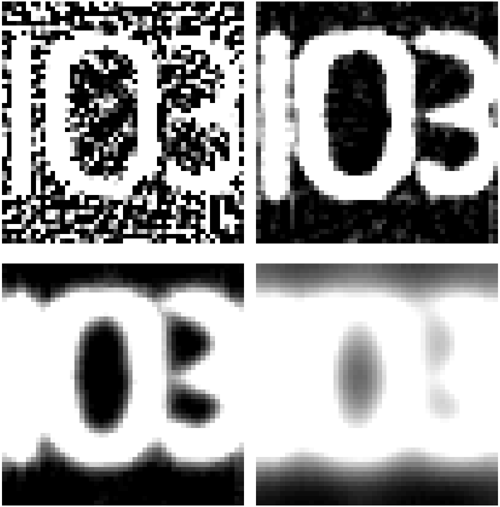
Reconstruction

reconstruction with $\lambda = 10$



Reconstruction

reconstructions with $\lambda = 10^{-6}, 20, 230, 2600$



Varying the number of line integrals

reconstruct with $m = 100, 400, 2500, 6400$ lines (with $\lambda = 10, 15, 25, 30$)