E209A LECTURE 11

GOALS OF THIS LECTURE:

- Lyapunov theory for asymptotic and exponential stability.
- Examples of Lyapunov.
- LaSalle's Theorem.
- Examples of LaSalle.

REFS:  
SASTRY § 5.3, 5.4, 5.5
KHALIL § 4.1, 4.2, 4.4, 4.5
LaSalle
**DEFN** \( x_e = 0 \) is said to be an **ASYMPTOTICALLY STABLE** equilibrium point of (NL) \( \dot{x} = f(x(t), t) \) if:

1. \( x_e = 0 \) is **STABLE** (equilibrium point)
2. There exists a \( S > 0 \) such that
   \[
   \| x_0 \| < S \Rightarrow \lim_{t \to \infty} \| x(t) \| = 0
   \]

**Remark:** need both (1) and (2)

(2) **does not** imply (1).

**Example:**
\[
\begin{align*}
\dot{x}_1 &= x_1^2 - x_2^2 \\
\dot{x}_2 &= 2x_1x_2
\end{align*}
\]

Identify \( x_1 = \infty \)
with \( x_1 = -\infty \).

This system is **not asymptotically stable** because it is not even stable. Given \( \epsilon > 0 \), there are always initial conditions close to \( x_2 = 0 \) which will exit the \( \epsilon \)-ball before converging to 0.
\text{Def} \ x_e = 0 \text{ is a } \underline{\text{Globally Asymptotically Stable}} \text{ equilibrium point of } (\text{NL}) \text{ if}

1. \ x_e = 0 \text{ is an asymptotically stable eq.}
2. \ \lim_{t \to \infty} x(t) = 0 \text{ for all } x_0 \in \mathbb{R}^n.

\text{(Note: Global Stability is defined in the obvious way.)}

\text{Def} \ x_e = 0 \text{ is an } \underline{\text{Exponentially Stable}} \text{ equilibrium point of } (\text{NL}) \text{ if there exists } m, \alpha > 0 \text{ such that}

\|x(t)\| \leq me^{-\alpha t} \|x_0\|

for all \ x_0 \in B_r, t > 0. \text{ The constant } \alpha \text{ is an estimate of (and is called) the rate of convergence.}

\text{(Note: Global Exponential Stability requires } x_0 \in \mathbb{R}^n).}

\text{Exponential Stability } \subset \text{ Asymptotic Stability } \subset \text{ Stability}
**NOTE 1:** The type of stability we are interested in depends on the engineering system at hand. For example, for a thermostat in a room we are usually satisfied with mere stability, whereas for a car's cruise control system, or an aircraft's autopilot system, we require exponential stability.

**NOTE 2:** Even if the system (NL) is stable (asymptotic, exponential...) the solution $x(t)$ need not be continuous anywhere except at $\tau_e = 0$. 
LYAPUNOV ASYMPTOTIC STABILITY THEOREM.

Consider \( \dot{x}(t) = f(x(t), t) ; x(t_0) = x_0 \) with equilibrium state \( x_e = 0 \).

If \( \exists r > 0 \) such that

1. \( V(x, t) = \text{p.d., decreasing, } L\text{-fn on } G_r \)
2. \( -\dot{V}(x, t) = \text{pd on } G_r \) [new condition]

Then \( x_e = \text{asymptotically stable} \).

**Intuition** [not a formal proof].

1. \( \Rightarrow x_e \text{ is stable (1st) by Lyapunov Stability Theorem.} \)

2. \( \Rightarrow \left[ \frac{dV}{dt}(x(t), t) = \dot{V} < 0 \text{ whenever } x(t) \neq 0 \forall t \geq t_0 \right] \)

\( \Rightarrow \left[ V(x(t), t) \rightarrow 0 \right] \Rightarrow \left[ x(t) \rightarrow 0 \right] \)

\( \uparrow V = \text{pd, decr.} \)

\( \Rightarrow x_e = \text{asymptotically stable (a.s)} \)

**Note:** Global theorems result from replacing \( G_r \) with \( \mathbb{R}^n \).
# Summary (Basic Lyapunov Theorems)

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Sometimes use the following notation:

- LPDF "locally PD function"
- PDF "PD function"
EXAMPLES:

(1). \[ \begin{align*}
\dot{x}_1 &= -x_2 + x_1 (x_1^2 + x_2^2 - 1) \\
\dot{x}_2 &= x_1 + x_2 (x_1^2 + x_2^2 - 1)
\end{align*} \]

Choose \( \nu(x) = x_1^2 + x_2^2 \) LPDF (by inspection)

\[ \dot{\nu}(x) = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1) \]

so \( -\dot{\nu}(x) \) is LPDF for \( \{x: x_1^2 + x_2^2 < 1\} \)

\( \Rightarrow 0 \) is a locally asymptotically stable equilibrium

(we knew this already by using this example before)

[NOTE: 0 is not globally asymptotically stable (abbr. G.A.S.) since there is a limit cycle of radius 1]
EXAMPLES

(2) \[ \begin{array}{c}
\begin{array}{c}
\text{Coulomb: } \dot{x}_1 = x_2 \\
\text{Faraday: } \dot{x}_2 = -f(x_2) - g(x_1)
\end{array}
\end{array} \]

\( x_1 \) change on capacitor
\( x_2 \) velocity through inductor
\( f(x_2), g(x_1) \) voltages.

Resistor \& Capacitor are \underline{locally Passive}:
\[ \begin{align*}
\quad & x f(x) \geq 0 \quad \forall x \in [-x_0, x_0] \\
\quad & x g(x) > 0 \quad \underline{local}
\end{align*} \]

- Use total energy of the system as a Lyapunov function candidate

\[ V(x) = \frac{x_2^2}{2} + \int_0^{x_1} g(\xi) d\xi \]

- \( V(x) \) is \{LPDF [provided that \( g(x_1) \) is not identically zero on some interval ]

\[ V(x) = x_2 [-f(x_2) - g(x_1)] + g(x_1) x_2 \]
\[ = -x_2 f(x_2) \leq 0 \]
\[ \Rightarrow \text{STABILITY} \quad [\text{of } (0,0)] \]
SYNCHRONOUS GENERATOR:

EXAMPLES: angle of rotor of generator

\[ \Theta \]

supply angle 0°

\[ P_m \rightarrow P_e \cdot \text{electrical power output} \]

mechanical power input

\[ M \cdot \text{moment of inertia of generator} \]

\[ D \cdot \text{generator's damping} \]

\[ B \cdot \text{susceptance of bus} \]

\[ \dot{\Theta} = \omega \]

\[ \dot{\omega} = -M^{-1}(DW + P_m - B\sin\Theta) \]

\[ = -M^{-1}DW - M^{-1}(P_m - B\sin\Theta) \]

\[ f(\omega) \quad g(\Theta) \]

equilibrium point at

\[ w_o = 0 \]

\[ \Theta_o = \sin^{-1}\frac{B}{P_m} \]

use same Lyapunov fn as before:

\[ V(\Theta, \omega) = \frac{1}{2}M\omega^2 + P_m \Theta + B\cos\Theta \]

Translated:

\[ V(\Theta, \omega) - V(\Theta_o, w_o) \]

is an LPDF around \((\Theta_o, w_o)\)

\[ \dot{V}(\Theta, \omega) = -DW^2 \leq 0 \]

\[ \Rightarrow \text{STABILITY of } (\Theta_o, w_o) \]
**LaSalle's Invariance Principle**

(For Time Invariant Systems)

From the Lyapunov Stability Theorem, we know that if \( V(x) \) is LPDF (PDF) and \( \dot{V}(x) \leq 0 \), for \( x \in \mathbb{R} \) (or \( x \in \mathbb{R}^n \)) then \( \dot{x} = f(x) \) is stable at 0 (globally stable).

However, we may still be able to prove asymptotic stability in this case using:

**LaSalle's Principle**:

Define \( \Omega_c := \{ x \in \mathbb{R}^n : V(x) < C \} \)
Suppose \( \Omega_c \) is bounded, and \( \dot{V} \leq 0 \) for all \( x \in \Omega_c \).
Define \( S := \{ x \in \Omega_c : V(x) = 0 \} \)
Let \( M \) be the largest invariant set in \( S \). Then, whenever \( x_0 \in \Omega_c \), \( x(t) \) approaches \( M \) as \( t \to \infty \).
\[ \mathcal{L}_c = \{ x \in \mathbb{R}^n : v(x) \leq 0 \} \]
\[ S = \{ x \in \mathcal{L}_c : v(x) = 0 \} \]

\( M \) is the largest invariant set in \( S \).

**Idea of Proof:** if \( x_0 \in \mathcal{L}_c, x(t) \in \mathcal{L}_c \ \forall t \).

Let \( c_0 = \lim_{t \to \infty} v(x(t)) \) (we know this exists since \( v(x(t)) \) is bounded below).

**Fact:** if a trajectory is completely enclosed within a bounded set, then the set of "limit points" that the trajectory can tend to (ie. equilibria, limit cycles) is bounded. Further, the trajectory approaches this limit set as \( t \to \infty \). [Wiggins pp 46-50]

Let \( L \) be the "limit set" of \( x(t) \). Then \( v(y) = c_0 \) for \( y \in L \), and \( v(y) = 0 \) for \( y \notin L \). Therefore \( L \subseteq S \). But \( L \subseteq M \) also since \( L \) is invariant. \( x(t) \to M \ \text{as} \ t \to \infty \).
How To Use Caselle's Principle to Establish Asymptotic Stability?

Caselle's Theorem (1960)

Given \( \dot{x} = f(x) \), \( v: \mathbb{R}^n \rightarrow \mathbb{R} \), I.P.D.F.

\[ \dot{v}(x) \leq 0 \text{ for } x \in \Omega_c \]

where \( \Omega_c = \{ x \in \mathbb{R}^n : v(x) \leq c \} \)

Let \( S = \{ x \in \Omega_c : \dot{v}(x) = 0 \} \) which stays

Then if the only trajectory in \( S \) is \( x(t) = 0 \), \( \dot{x} = f(x) \) is locally asymptotically stable near \( x^* = 0 \).

Remarks:

1) Global version of above:

(Uses \( v(x) \): \( \mathbb{R}^n \rightarrow \mathbb{R} \) PDF

\[ \dot{v}(x) \leq 0, \text{ } x \in \mathbb{R}^n \]

\[ \Rightarrow \text{Global Asymptotic Stability} \]

2) The theorem works because, even though \( \dot{v}(x) \leq 0 \), the only trajectory of \( \dot{x} = f(x) \) with \( \dot{v}(x(t)) = 0 \) is \( x(t) = 0 \) (trivial trajectory).
EXAMPLES

1) \begin{align*}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= -f(x_2) - g(x_1)
\end{align*}

- Resistor: Capacitor are locally Passive
  \( x f(x) \geq 0, \quad x g(x) \geq 0 \quad \forall x \in [-x_0, x_0] \)

- As before, \( V(x) = \frac{x_2^2}{2} + \int_0^{x_1} g(s) ds \)
  \( V(x) \) is Lyapunov [Assuming no deadband in \( g(\cdot) \)]
  \( \dot{V}(x) = -x_2 f(x_2) \leq 0 \)
  \( \Rightarrow \) STABILITY of \((0,0)\).

CAN WE SAY MORE?

Let \( D = \{ x \in \mathbb{R}^2 \mid -x_0 < x_i < x_0 \} \quad i = 1, 2 \)
\( S = \{ x \in D \mid \dot{V}(x) = 0 \} \)
To characterize \( S \), note that
\( \dot{V}(x) = 0 \Rightarrow x_2 f(x_2) = 0 \)
\( \Rightarrow x_2 = 0 \), since \(-x_0 < x_2 < x_0\).
EXAMPLES

1) (cont'd)

\[ S = \{ x \in \mathbb{D} / x_2 = 0 \} \]

Now suppose \( x(t) \) is a trajectory that stays in \( S \):

\[ x_2 = 0 \Rightarrow \dot{x}_2 = 0 \Rightarrow g(x_1(t)) = 0 \Rightarrow x_1(t) = 0 \]

Therefore, the only solution that can stay in \( S \) is \( x(t) = 0 \).

Therefore, \( x^* = 0 \) is asymptotically stable.
EXAMPLES

2). Consider the system
   \[ y' = ay + u \quad y, u \in \mathbb{R} \]
   \( u \) is called the "control" and we can manipulate it to make the system do what we want.

Consider the adaptive control law
   \[ u = -ky, \quad k = \gamma y^2, \quad \gamma > 0 \]

The closed loop system may be written as:
   \[ y' = -(k-a)y \]
   \[ k = \gamma y^2 \]

- \( y = 0 \) is a line of equilibria

We want to show that the trajectory of the system approaches this equilibrium set as \( t \to \infty \), meaning that the adaptive controller succeeds in regulating \( y \) to zero.
Consider the Lyapunov function candidate

\[ V(y, \kappa) = \frac{1}{2} y^2 + \frac{1}{2\gamma} (k-b)^2 \]

where \( b > a \).

\[ V(y, \kappa) = -y^2(k-a) + y^2(k-b) = -y^2(b-a) \leq 0. \]

For any finite \( c > 0 \), the set

\[ \Omega_c = \{ [y, \kappa] \in \mathbb{R}^2 \mid V(y, \kappa) \leq c \} \]

is positively invariant bounded.

\[ S = \{ [y, \kappa] \in \Omega_c \mid y = 0 \} \]

\[ M = S. \]

From LaSalle's Theorem, every trajectory starting in \( \Omega_c \), \([y, \kappa](t)\), approaches \( M \) as \( t \to \infty \) [This actually holds globally]

\[ \therefore y(t) \to 0 \text{ as } t \to \infty. \]
Global version of LaSalle's Theorem

Consider the system

\[ \dot{x} = f(x) \quad , \quad x_0 = 0 \]

let \( v(x) \) be a PDF with \( \dot{v} \leq 0 \quad \forall x \in \mathbb{R}^n \)

If the set

\[ S = \{ x \in \mathbb{R}^n : v(x) = 0 \} \]

contains no invariant sets other than the origin, then the origin is globally asymptotically stable.

LaSalle's Theorem for Periodic Systems

Consider the system

\[ \dot{x} = f(x, t) \quad , \quad x_0 = 0 \]

where \( f \) is periodic

\[ f(x, t) = f(x, t+T) \quad \forall t \in \mathbb{R}^n \]

Further, let \( v(x, t) \) be a PDF which is periodic in \( t \) with period \( T \). Define

\[ S = \{ x \in \mathbb{R}^n : v(x, t) = 0 \quad \forall t \geq 0 \} \]

Then if \( \dot{v}(x, t) \leq 0 \quad \forall x \in \mathbb{R}^n \quad \forall t \geq 0 \) and \( S \) contains no invariant sets other than the origin, then the origin is globally a.s.
Generalization of LaSalle's Theorem:

A difficulty arises in extending LaSalle's Theorem to arbitrary time-varying systems, which is that

\[ \{ x : \dot{v}(x,t) = 0 \} \]

may be a time-varying set.

However, if we can assume that:

\[ \dot{v}(x,t) \leq -W(x) \leq 0 \]

Then the set \( S \) may be defined as:

\[ \{ x : W(x) = 0 \} \]

and LaSalle's Theorem may be generalized as follows:

Consider \( \dot{x} = f(x,t), \ x_e = 0 \), and suppose that for \( x \in B_r \) (a ball of radius \( r \) around \( x_e \)), there exists a function \( v(x,t) \) such that for functions \( x_1, x_2 \) of class \( K \):

\[ x_1(1x1) \leq v(x,t) \leq x_2(1x1) \]

Also, assume that for some
non-negative function \( w(x) \)

\[
\dot{u}(x,t) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} f(x,t) \leq -w(x) \leq 0
\]

Then, for all \( \|x(t_0)\| \leq \alpha^{-1}_2(x,(r)) \), the trajectories \( x(t) \) are bounded and

\[
\lim_{t \to \infty} w(x(t)) = 0
\]

[ meaning that \( x(t) \) approached a set \( E \) defined by:

\[
E := \{ x \in B_r : w(x) = 0 \}.
\]
Lyapunov Theory. Example.

Nonlinear Spring:

\[ M \ddot{x} = -F_s \quad \text{where} \quad F_s = k(x) \]

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
-\frac{1}{M} k(x_1)
\end{bmatrix}
\]

Example: Duffing equation, no friction

Nonlinear spring \( F_s = k(x) \)

Restoring: \( k(x)x \geq 0 \) for all \( x \)
\[ V(x) = \frac{1}{2} mx_2^2 + \int k(y) \, dy \]

total energy. K.E P.E.

\[ v(x) = [k(x_1), mx_z] [\begin{bmatrix} x_2 \\ -\frac{1}{m} k(x_1) \end{bmatrix}] = 0 \]

\[ v(x) \leq 0. \]

So if \( v(x) \) is LPD or PD we can determine the stability of the system \( x_1 \).

\[ V(x) = \frac{1}{2} mx_2^2 + \int k(y) \, dy \]

\[ \geq 0 \]

\[ \geq 0 \] since restoring spring (ie \( x_1, k(x_1) \geq 0 \))

\[ \geq 0 \]

\[ v(x) \geq 0. \]

\[ (\text{semi-PD}) \]

When is \( V(x) \) LPD or PD? When \( x_1 \cdot k(x_1) \geq 0 \) and when the spring has no deadzone at 0. i.e. cannot have this situation:

\[ k(x) = 0 \text{ when } x = 0 \]