E209A LECTURE 13

Goals:
- Introduce nonlinear control: linearization by state feedback.
- Relative degree
- Overall control topology.

Refs
Sastry G 9.1, 9.2
Khalil Ch. 13.
Linearization by State Feedback:


Main Idea: Given a nonlinear system of the form
\[
\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{align*}
\]
not as general as \( \dot{x} = f(x,u) \) as \( y = h(x) \)
use (nonlinear) state feedback \( u = k(x) + v \) to make the system \( \dot{x} \rightarrow y \) \( y \rightarrow y \)
EXACTLY linear
- not the same as Jacobian linearization and approx. linear.
EXAMPLE 1. Linear, Time-Invariant Single-Input Single-Output (SISO)

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -x_4 x_4 - x_3 x_3 - x_2 x_2 - x_1 x_1 + bu \\
y &= c x_1 & \alpha_i, c \in \mathbb{R}.
\end{align*}
\]

a) Find a memoryless state feedback of the form \( u = k x + v \) (new input) so that all of the eigenvalues are at \( s = -1 \).

SOLUTION:

\[
\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ b \end{bmatrix} u
\]

\( \Rightarrow \) system is controllable for \( b \neq 0 \) (check rank \( [B | AB | A^2 B | A^3 B] \))
For all eigenvalues at \( s = -1 \) in the closed loop system, we must have
\[
\bar{A} = A + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}
\]
where \( \bar{A} \) has characteristic polynomial:
\[
(s+1)^4 = s^4 + 4s^3 + 6s^2 + 4s + 1
\]
\[
\therefore \bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -4 \end{bmatrix} \]
\[
\beta_1 = -\alpha_1 + bk_1 \\
\beta_2 = -\alpha_2 + bk_2 \\
\beta_3 = -\alpha_3 + bk_3 \\
\beta_4 = -\alpha_4 + bk_4
\]
\[
k_1 = \frac{(\alpha_1 - 1)}{b}, \quad k_2 = \frac{(\alpha_2 - 4)}{b}, \quad k_3 = \frac{(\alpha_3 - 6)}{b}, \quad k_4 = \frac{(\alpha_4 - 4)}{b}
\]

b) Now suppose that the equation for \( x_4 \) in the original system is replaced by \( \dot{x}_4 = \gamma(x_1, x_2, x_3, x_4) + bu \) where \( \gamma(\cdot, \cdot, \cdot, \cdot) \) is known, nonlinear. Use a memoryless state feedback of the form
\[
U = K(x) + \gamma
\]
\[ \text{nonlinear} \]
... So as to make the input-output transfer function from \( u \) to \( y \)

\[
\frac{ce}{(s+1)^4}.
\]

**SOLUTION:**

\[
\dot{x_4} = y_4(x_1, x_2, x_3, x_4) + b u
\]

choose \( u = -\frac{\nu}{b} (x_1, x_2, x_3, x_4) + k x + \nu \)

\[
\therefore \dot{x}_4 = b K x + b \nu \quad \text{linear in } x, \nu!
\]

\[
\therefore \bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ b k_1 & b k_2 & b k_3 & b k_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ b \end{bmatrix} u
\]

\[
\therefore \text{here } k_1 = -\frac{1}{b}, k_2 = -\frac{4}{b}, k_3 = -\frac{6}{b}, k_4 = -\frac{4}{b}.
\]

**Check:** The transfer function from \( u \) to \( y \):

\[
y = c x,
\]

now \( \dot{x}(t) = \bar{A} x(t) + B u(t) \) where \( B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b \end{bmatrix} \)

\[
(3I - \bar{A}) x(s) = B \nu(s)
\]

\[
\therefore x(s) = (3I - \bar{A})^{-1} B \nu(s)
\]

and \( y(s) = c x(s) = [c \ 0 \ 0 \ 0 \ 0] x(s) \)
\[
\begin{align*}
\therefore \quad \frac{y(s)}{u(s)} :& = G(s) = \begin{bmatrix} c & 0 & 0 & 0 \end{bmatrix} (sI - A)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \\
& = \frac{cb}{(s+1)^4}
\end{align*}
\]

**EXAMPLE 2:** Consider the following single-input, single-output nonlinear system:

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
\dot{\xi}_3 &= \xi_4 \\
\vdots & \quad \vdots \\
\dot{\xi}_r &= a(\xi, \eta) + b(\xi, \eta) u \\
\eta &= q(\xi, \eta).
\end{align*}
\]

Output: \( y = \xi_1 \)

\( u \in \mathbb{R} \), \( y \in \mathbb{R} \), \( \xi := \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_r \end{bmatrix} \in \mathbb{R}^r \), \( \eta \in \mathbb{R}^{n-r} \)

and \( a(\xi, \eta), b(\xi, \eta), q(\xi, \eta) \) are all smooth, scalar functions of \( \xi \) and \( \eta \):

ie. \( a(\cdot, \cdot) : \mathbb{R}^r \times \mathbb{R}^{n-r} \rightarrow \mathbb{R} \).

Also, \( b(\xi, \eta) \neq 0 \).
Consider the state feedback law
\[ u = -\frac{a(x, \eta) + \nu}{b(x, \eta)} \]

With this state feedback law, find the relationship between \( y \) and \( u \). What is remarkable about it? Is the closed loop system observable?

Solution:
\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_r &= a(x, \eta) + b(x, \eta) \left[ -\frac{a(x, \eta) + \nu}{b(x, \eta)} \right] \\
\dot{\eta} &= q(x, \eta) \\
y &= x_1 
\end{align*} \]

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_r &= \nu \\
\dot{\eta} &= q(x, \eta) \\
y &= x_1 
\end{align*} \]

\[ \text{We note that:} \]
\[ \begin{align*}
(1) \quad \eta \text{ does not affect } x. \\
(2) \quad \text{we can compute } \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{1}{s^r} \\
\end{align*} \]

\[ \text{ie. the feedback law above was chosen so that the i/o relationship is described by a linear system!} \]

Now, suppose \( u(t) = 0, \quad x_1(0) = x_2(0) = \ldots = x_r(0) = 0 \) and \( \eta(0) \neq 0 \).
from the above, we obtain:
\[ y(t) = 0 \quad \forall t > 0 \]

Therefore, the states \( \eta \) are not observable.

**Input-Output Linearization for SISO Systems**

Given
\[
\begin{align*}
\dot{x} &= f(x) + g(x) \ u \\
y &= h(x)
\end{align*}
\]

with \( x \in \mathbb{R}^n \), \( u \in \mathbb{R} \), \( y \in \mathbb{R} \)

\( f(x) \) smooth ("infinitely differentiable"
\( g(x) \) smooth function from \( \mathbb{R}^n \) to \( \mathbb{R} \).

Let \( x^* \) be an equilibrium point of the undriven \((u=0)\) system, i.e. \( f(x^*) = 0 \).

Let the following calculations be for \( x \in U = B_r(x^*) \)

Differentiating \( y \) with respect to time:
\[
\dot{y} = \frac{\partial h}{\partial x} \dot{x}
\]
\[
= \frac{\partial h}{\partial x} \left[ f(x) + g(x) u \right]
\]
\[
= \frac{\partial h}{\partial x} f(x) + \frac{\partial h}{\partial x} g(x) u \quad \text{(*)}
\]
We've seen this type of derivative before (Lyapunov Theory):

\[
\frac{\partial h}{\partial x} \cdot f(x) =: L_f h(x)
\]

"Lie derivative of \( h \) with respect to \( f \)"

\[
\frac{\partial h}{\partial x} \cdot g(x) =: L_g h(x)
\]

"Lie derivative of \( h \) with respect to \( g \)"

\[\bullet\text{ if } |L_g h(x)| > \delta, \delta > 0 \text{ "bounded away from zero" for all } x \in U, \text{ then the state feedback law is given by} \]

\[
U = \frac{1}{L_g h(x)} \left[-L_f h(x) + u\right]
\]

\[\bullet\text{ if } L_g h(x) = 0, \text{ meaning } L_g h(x) = 0 \forall x \in U, \text{ we differentiate (*) again:} \]

\[
\ddot{y} = \frac{\partial L_f h}{\partial x} f(x) + \frac{\partial^2 L_f h}{\partial x^2} g(x) u
\]

\[=: L^2_f h(x) + L_g L_f h(x) u\]
• if \(|\log f h(x)| > S_2\), \(S_2 > 0\) \(\forall x \in U\) then the state feedback law is given by

\[ u = \frac{1}{\log f h(x)} \left( -\log f h(x) + v \right) \]

ALGORITHM:

given \( \dot{x} = f(x) + g(x) \, u \)
\( y = h(x) \)

• differentiate \( y \) with respect to time.

\( \dot{y} = \log f h(x) + \log h(x) \, u \)

• if \(|\log h(x)| > S_1\) for all \( x \in U \), let

\[ u = \frac{1}{\log h(x)} \left[ -\log f h(x) + v \right] \]

I/O linear system is \( \dot{y} = v \)

• if \( \log h(x) \equiv 0 \), differentiate again

\( \ddot{y} = \log f^2 h(x) + \log f h(x) \, u \)

• if \(|\log f h(x)| > S_2\) for all \( x \in U \), let

\[ u = \frac{1}{\log f h(x)} \left[ -\log f^2 h(x) + v \right] \implies \dot{y} = v \]

• if \( \log f h(x) \equiv 0 \), differentiate again.
More generally, if $\gamma$ is the smallest integer for which $\log f^i h(x) = 0$ on $U$ for $i = 0, \ldots, \gamma - 1$ and $\log f^{\gamma - 1} h(x)$ is bounded away from zero on $U$, then the control law given by

$$U = \frac{1}{\log f^{\gamma - 1} h(x)} (-g f^\gamma h(x) + v)$$

yields the $\gamma$th-order linear system from input $v$ to output $y$:

$$y^\gamma = v$$

**Defn** Strict Relative Degree

The SISO nonlinear system

$$\dot{x} = f(x) + g(x) v$$

$$y = h(x)$$

is said to have **strict relative degree** $\gamma$ at $x^* \in U$ if:

- $\log f^i h(x) = 0 \ \forall x \in U, \ i = 0, \ldots, \gamma - 2$
- $\log f^{\gamma - 1} h(x^*) \neq 0$
Overall block diagram:

\[ u = k(x, u) \]

\[ \dot{x} = f(x) + g(x)u \]

\[ h(x) \rightarrow y \]

linearization loop
Consider the two-input, two-output case:

\[
\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2
\]

\[
(y_1, y_2) = (h_1(x), h_2(x))
\]

\[
x \in \mathbb{R}^n
\]

\[
u = [u_1, u_2] \in \mathbb{R}^2
\]

\[
y = [y_1, y_2] \in \mathbb{R}^2
\]

Start with \( y_1 \): differentiate \( y_1 \) until an input appears, let \( \gamma_1 \) be the smallest integer such that at least one of the inputs appears in \( y_1^{(\gamma_1)} \leftarrow \gamma_1 \)th derivative of \( y_1 \).

Repeat for \( y_2 \), with \( \gamma_2 \) the smallest integer such that at least one of the inputs appears in \( y_2^{(\gamma_2)} \).

Thus,

\[
y_1^{(\gamma_1)} = L_{\gamma_1} f h_1 + L_{\gamma_1} g_1 L_{\gamma_1} f h_1 u_1 + L_{\gamma_1} g_2 L_{\gamma_1} f h_1 u_2
\]

\[
y_2^{(\gamma_2)} = L_{\gamma_2} f h_2 + L_{\gamma_2} g_1 L_{\gamma_2} f h_2 u_1 + L_{\gamma_2} g_2 L_{\gamma_2} f h_2 u_2
\]
let \( A(x) = \begin{bmatrix} Lg_1 L_f^{x_1-1} h_1 & Lg_2 L_f^{x_1-1} h_1 \\ Lg_1 L_f^{x_2-1} h_2 & Lg_2 L_f^{x_2-1} h_2 \end{bmatrix} \)

**Definition Vector Relative Degree**

The system \((\text{NL}-\text{TITO})\) is said to have vector relative degree \( \gamma_1, \gamma_2 \) at \( x^* \) if

\[
L g_i L f^k h_i(x) = 0 , \quad 0 \leq k \leq \gamma_i - 2
\]

for \( i = 1, 2 \) and the matrix \( A(x_0) \) is nonsingular.

Then, the state feedback law

\[
u = - A^{-1}(x) \begin{bmatrix} L f^{\gamma_1} h_1 \\ L f^{\gamma_2} h_2 \end{bmatrix} + A^{-1}(x) \nu
\]

yields a linear closed loop system

\[
\begin{bmatrix} y_1 \delta_1 \\ y_2 \delta_2 \end{bmatrix} = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}
\]