E209A LECTURE 15

GOALS:
- Worked example in I/O linearization
- Linearization in the Presence of Uncertainty
  - matching conditions
  - sliding mode control
  - worked example

REFS: SAstry §9.4, §9.5
Worked Problem: FB Linearization of the VTOL aircraft.

We know from class that for $n$-dimensional two-input two-output (TITO) systems, if the vector relative degree with respect to output $y = (y_1, y_2)$ is $(\gamma_1, \gamma_2)$, then use of the input/output linearizing control law for a desired output which is identically zero will result in $n - \gamma_1 - \gamma_2$ dimensional zero dynamics. If these zero dynamics are unstable, then use of the input/output linearizing control law may cause some of the state variables to become unstable.

One interesting alternative scheme is to select a different output $\bar{y} = (\bar{y}_1, \bar{y}_2)$ about which the system is exactly linearizable, i.e. about which the vector relative degree is $(\gamma_1, \gamma_2)$ where $\gamma_1 + \gamma_2$ is equal to the dimension of the state. Thus, if we linearize the system about this new output $\bar{y}$, there will be no zero dynamics left over and we'll be able to completely control (and therefore stabilize, regulate, track) all of the system dynamics, through the input/output linearizing control law.

Consider the following simplified model of the lateral dynamics of a Vertical Take-Off and Landing (VTOL) aircraft.

\[ \begin{align*}
\ddot{x} &= -u_1 \sin \theta + \varepsilon u_2 \cos \theta \\
\ddot{z} &= u_1 \cos \theta + \varepsilon u_2 \sin \theta - g \\
\dot{\theta} &= \lambda u_2
\end{align*} \]  

(aircraft is flying into the page)

Here, $x$ is the lateral position ($\dot{x}$, $\ddot{x}$ the lateral velocity, acceleration respectively), $z$ is the vertical position ($\dot{z}$, $\ddot{z}$ the vertical velocity, acceleration respectively), all with respect to the inertial frame. $(x_0, z_0)$ represent the body frame coordinates of the aircraft. $\theta$ is the roll angle ($\dot{\theta}, \ddot{\theta}$ the angular velocity, acceleration respectively). $u_1$ is the vertical thrust, $u_2$ represents the bleed air forces at wingtips. Also, $g$ is the gravitational acceleration, $\lambda$ is a term involving the moment of inertia of the aircraft, and $\varepsilon$ represents a small coupling between the wingtip forces and the vertical and lateral acceleration.

If we choose the output $y = (x, z)$, the system with respect to this output has vector relative degree $(2, 2)$, and the resulting zero dynamics in $(\theta, \dot{\theta})$ are unstable.

Here, I would like you to exactly linearize the system about new output $\bar{y} = (\bar{y}_1, \bar{y}_2)$ where

\[ \begin{align*}
\bar{y}_1 &= x + a \cos \theta - b \sin \theta \\
\bar{y}_2 &= x + a \sin \theta + b \cos \theta
\end{align*} \]  

where $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are design parameters to be chosen. This new output represents a point which is affixed to the body frame of the aircraft.

To simplify this task assume that $a = 0$, and choose $b$ (in the process of differentiating) so that the VTOL aircraft given by (1) is exactly linearizable with respect to $\bar{y} = (\bar{y}_1, \bar{y}_2)$. (HINTS: The new vector relative degree will be such that $\gamma_1 = \gamma_2$. You will need to define two new states $(\phi, \psi)$ in the process of differentiating, where

\[ \psi := u_1 - b \dot{\theta}^2 \]  

Thus, you should find that your final linearized system has 8 state variables (where $u_1$ and $\dot{u}_1$ are now included in the state as shown above), new input $(\bar{u}_1, u_2) = (\psi, u_2)$, and no zero dynamics.)
\begin{align*}
\ddot{y}_1 &= x + a \cos \theta - b \sin \theta \\
\ddot{y}_2 &= \varepsilon + a \sin \theta + b \cos \theta
\end{align*}

\begin{align*}
a = 0 &\Rightarrow \ddot{y}_1 = x - b \sin \theta \\
\ddot{y}_2 &= \varepsilon + b \cos \theta
\end{align*}

\begin{align*}
\dddot{y}_1 &= \ddot{x} - b \dot{\theta} \cos \theta \\
\dddot{y}_2 &= \ddot{\varepsilon} - b \dot{\theta} \sin \theta
\end{align*}

\begin{align*}
\dddot{y}_1 &= \ddot{x} - b \left[ \dot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta \right] \\
\dddot{y}_2 &= \ddot{\varepsilon} - b \left[ \dot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta \right]
\end{align*}

Now \ \ddot{x} = -u_1 \sin \theta + \varepsilon u_2 \cos \theta \\
\ddot{\varepsilon} = u_1 \cos \theta + \varepsilon u_2 \sin \theta - g. \\
\dot{\theta} = \lambda u_2

\begin{align*}
\dddot{y}_1 &= -u_1 \sin \theta + \varepsilon u_2 \cos \theta - b \left[ \dot{\lambda} u_2 \cos \theta - \dot{\theta}^2 \sin \theta \right] \\
&= -\gamma \sin \theta + (\varepsilon - b \lambda) u_2 \cos \theta \\
\dddot{y}_2 &= u_1 \cos \theta + \varepsilon u_2 \sin \theta - g - b \left[ \dot{\lambda} u_2 \sin \theta + \dot{\theta}^2 \cos \theta \right] \\
&= \gamma \cos \theta + (\varepsilon - b \lambda) u_2 \sin \theta - g.
\end{align*}

Now if we were to stop here
\begin{equation*}
\begin{bmatrix}
\dddot{y}_1 \\
\dddot{y}_2
\end{bmatrix} = 
\begin{bmatrix}
-\gamma \\
\gamma
\end{bmatrix} 
+ 
\begin{bmatrix}
0 \\
0
\end{bmatrix} \begin{bmatrix}
\varepsilon - b \lambda \\
\varepsilon - b \lambda
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\end{equation*}

ie. the inputs do not appear independently.
set $b = \frac{\gamma}{A}$ to make $u_2$ disappear.

\[ a = 0, \quad b = \frac{\gamma}{A} \quad \text{and} \quad \begin{aligned}
\dot{y}_1 &= -\gamma \sin \theta \\
\dot{y}_2 &= \gamma \cos \theta.
\end{aligned} \]

Now continue:

\[ \begin{aligned}
y_1^{(3)} &= -\gamma \sin \theta - \gamma \cos \theta \\
y_2^{(3)} &= \gamma \cos \theta - \gamma \sin \theta.
\end{aligned} \]

\[ \begin{aligned}
y_1^{(4)} &= -\gamma \sin \theta - \gamma \cos \theta - \gamma \cos \theta - \gamma \sin \theta \\
&= -\gamma \sin \theta - \gamma \cos \theta + \gamma \cos \theta + \gamma \sin \theta \\
&= -\gamma \sin \theta - 2\gamma \cos \theta + \gamma \cos \theta - \gamma \sin \theta \\
&= -\gamma \sin \theta - 2\gamma \cos \theta + \gamma \cos \theta - \gamma \sin \theta.
\end{aligned} \]

\[ \begin{aligned}
y_2^{(4)} &= \gamma \cos \theta - \gamma \sin \theta - \gamma \sin \theta - \gamma \cos \theta \\
&= \gamma \cos \theta - 2\gamma \sin \theta - \gamma \cos \theta - \gamma \sin \theta.
\end{aligned} \]

Thus, \[ \begin{aligned}
\dot{y}_1 &= \gamma - \frac{\gamma}{A} \sin \theta \\
\dot{y}_2 &= 0 + \frac{\gamma}{A} \cos \theta.
\end{aligned} \]

new input \[ \begin{aligned}
\{ u_1 &= \gamma \\
u_2 \}_{\text{new}}
\end{aligned} \]

status $x, \dot{x}, z, \ddot{z}, e, \dot{e}, \gamma, \dot{\gamma}$ no more dynamic!
Linearization in the Presence of Uncertainty

\[ x = f(x) + \Delta f(x) + g(x) u + \Delta g(x) u \]
\[ y = h(x) \]

unperturbed \( \Delta f(x) = 0, \Delta g(x) = 0 \)
Assume the unperturbed system has strict relative degree \( k \).

For the unperturbed system, we would have:
\[ \dot{x}_1 = h \]
\[ \dot{x}_2 = \dot{h}h \]
\[ \vdots \]
\[ \dot{x}_k = \dot{h}x^{k-1}h \]
\[ \dot{x}_k = b(\xi, \eta) + a(\xi, \eta) u \]
\[ \dot{u} = q(\xi, \eta) \]

and we'd choose the feedback linearizing control law:
\[ U = \frac{1}{\log f(x)} \left[ -L^* h(x) + v \right] \]

Now suppose \( \Delta f \) and \( \Delta g \) are present:
\[ \dot{x}_1 = \frac{\partial h}{\partial x} \cdot \dot{x} = \frac{\partial h}{\partial x} \left[ f + \Delta f + g \dot{u} + \Delta g u \right] \]
\[ = \dot{x}_2 + L^* f h + L^* g h u. \]
Similarly

\[ \dot{x}_2 = \dot{x}_3 + \log f(y; h) + \log (L_f h) u. \]

\[ \vdots \]

\[ \dot{x}_n = b(x, u) + \log f(y; h) + a(x; u) u + \log (L_f^{-1} h) u. \]

and

\[ \ddot{y} = g(x, u) + \log f h u. \]

Under what conditions does the linearizing control law for the unperurbed system \( u(*) \) linearize the perturbed system?

**IF** the perturbations \( \Delta f \) and \( \Delta g \) satisfy

\[ \log L_f^i h = 0 \text{ for } 0 \leq i \leq \chi - 1 \]

\[ \log L_f^{-1} h = 0 \text{ for } 0 \leq i \leq \chi - 1 \]

Then the linearization scheme is robust in the presence of these uncertainties \( \Delta f, \Delta g \).
These conditions are known as the matching conditions and may be written as:

\[
\begin{bmatrix}
\frac{dh}{df} \\
\vdots \\
\frac{dh}{df^{n-1}}
\end{bmatrix} \Delta f = 0 ; \begin{bmatrix}
\frac{dh}{df} \\
\vdots \\
\frac{dh}{df^{n-1}}
\end{bmatrix} \Delta g = 0
\]

or stated as

"The relative degree of the disturbance is greater than the relative degree of the non-disturbed terms."

Remark: It is important to notice that the \([n]\) variables associated with the zero dynamics may be perturbed... in particular, the zero dynamics may be destabilized by the perturbation.
Consider
\[ \dot{x} = f(x) + \Delta f(x) + g(x) u + \Delta g(x) u \]
\[ y = h(x) \]
and suppose
\[ \begin{bmatrix} \frac{dh}{dy} h \\ \frac{df}{dy} h \\ \frac{d^2 h}{dy^2} h \end{bmatrix} \Delta g = 0 \]
but that
\[ \begin{bmatrix} \frac{dh}{dy} h \\ \frac{df}{dy} h \\ \frac{d^2 f}{dy^2} h \end{bmatrix} \Delta f = 0 \]
A weaker than matching conditions.

ie. \( \text{rel} \text{deg}(\Delta f) = 0 \) (relative degree of disturbance \( \Delta f \) is \( \sigma \), and relative degree of \( \Delta g > \sigma \)).

**IDEA:**
We would like to devise a tracking control law for desired trajectory \( y_d(t) \) with as few assumptions as possible on the perturbation vector fields: \( \Delta f(x), \Delta g(x) \).
Under these conditions:
\[ \ddot{\xi}_1 = \ddot{\xi}_2 = \ddot{\xi}_3 \]
\[ \ddot{\xi}_n = b(\xi, n) + \Delta b(\xi, n) + a(\xi, n)u \]
\[ \dot{n} = q(\xi, n) + \Delta q(\xi, n) + \Delta p(\xi, n)u \]

where  
\[ \Delta b(\xi, n) = L_0 f l_f^{-1} h \]
\[ \Delta q(\xi, n) = L_0 f n \]
\[ \Delta p(\xi, n) = L_0 g n \]

Task: get \( y(t) \rightarrow y_{d}(t) \)

Assume \( |\Delta b(\xi, n)| \leq K \) and define a "sliding surface" \( S(x, t) \) by:
\[ e_0(t) = y_d(t) - y(t) \]
\[ S(x, t) = e_0^\gamma - 1 + \alpha_{\gamma-2} e_0^{\gamma-2} + \cdots + \alpha_0 e_0 \]
\[ = (y_d - l_f^{-1} h) + \alpha_{\gamma-2} (y_d - l_f^{-1} h) + \cdots + \alpha_0 (y_d - h) \]

where the \((\gamma-1)\) \( \alpha_i \)'s are chosen so that
\[ \left( S^\gamma - 1 + \alpha_{\gamma-2} S^{\gamma-2} + \cdots + \alpha_0 \right) = 0 \]
is Hurwitz (all poles in CLHP)
Idea: if we can get the state of the system \( x \) to the surface \( \mathbb{E} \) \( \mathbb{E} x: S(x,t) = 0 \), then we are guaranteed that the output error goes to zero.

Choose

\[
U = \frac{1}{\log f^{-1} h} \left[ y_d^{(r)} - L f^r h + \alpha_{r-2} e_0^{r-1} + \ldots + \alpha_0 e_0 - 1.1 K \text{sgn} S(x,t) \right]
\]

an exact linearizing control law

\[
U = \frac{1}{\log f^{-1} h} \left[ -L f^r h(x) + v \right]
\]

where \( v = y_d^{(r)} + \alpha_{r-2} e_0^{r-1} + \ldots + \alpha_0 e_0 - 1.1 K \text{sgn} S(x,t) \)

Why do we use this form of \( v \)?

Compute \( \frac{d}{dt} S(x,t) \) (with \( U \) as given above) to find that:

\[
\frac{d}{dt} S(x,t) = -K - 1.1 K \text{sgn} S(x,t) \\
\leq -0.1 K \text{sgn} S(x,t)
\]

ie. When \( S(x,t) > 0 \), \( \dot{S} \leq -0.1 K \) made attractive.

\( S(x,t) < 0 \), \( \dot{S} \geq 0.1 K \) makes \( x: S(x,t) = 0 \)
Uncertainty and Sliding Mode (cont'd)

worked example  Sliding mode control of uncertain linear systems.

Consider the linear time-varying control system, with $x \in \mathbb{R}^3$:

$$
\dot{x} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-a_1(t) & -a_2(t) & -a_3(t)
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} u
$$

The $a_i(t)$ are not known exactly, we only know that $\alpha_i \leq a_i(t) \leq \beta_i$ for $i = 1, 2, 3$ for known $\alpha_i, \beta_i$. The goal is to get the output of the system $y(t) = x_1(t)$ to track a specified 3-times differentiable function $y_d(t)$, where $y_d^{(3)}(t)$ is assumed bounded. We define a sliding surface $S_0 := \{x \in \mathbb{R}^3 : s(x, t) = 0\}$, where

$$
s(x, t) = (x_2 - y_d^{(2)}(t)) + c_1(x_2 - y_d(t)) + c_2(x_1 - y_d(t))
$$

We will choose a control law of the form

$$
u = y_d^{(3)}(t) + \gamma_1(x)x_1 + \gamma_2(x)x_2 + \gamma_3(x)x_3 + k_1(x,t)(x_2 - y_d(t)) + k_2(x)(x_2 - y_d^{(2)}(t)) - k_3 \text{sgn}(s(x, t))\tag{16}
$$

where $\text{sgn}(s(x, t))$ stands for the sign of $s(x, t)$, defined to be $+1$ when $s(x, t) > 0$ and $-1$ when $s(x, t) < 0$. I would like you to choose feedback functions $\gamma_i(x)$ and $k_3$ so as to make $S_0$ a sliding surface; i.e., choose $u$ so that the system (14) satisfies the global sliding condition:

$$
\frac{d}{dt}s^2(x, t) \leq -\varepsilon |s(x, t)|
$$

for some $\varepsilon$. In your analysis, choose $k_1(x, t) = -c_2$, $k_2(x, t) = -c_1$.

Show that the system reaches the sliding surface in finite time. Now, show that once on the sliding surface, we can make the tracking error $e(t) = x_1(t) - y_d(t)$ go to zero asymptotically by a clever choice of the $c_i$.

How would you expect the control law to perform if the bounds $(\alpha_i, \beta_i)$ are close together (little uncertainty) versus if the bounds are far apart (high uncertainty)?

Solution

we first need to choose $\gamma_i(x)$ and $k_3$ so that

$$
\frac{d}{dt}s^2(x, t) \leq -\varepsilon |s(x, t)|
$$
\[ \frac{d}{dt} s^2 = 2 \dot{s} \cdot s \]
\[ = 2s \left( \dot{x}_3 - y_d^{(3)} + c_1 (x_3 - y_d^{(2)}) + c_2 (x_2 - y_d) \right) \]
\[ = 2s \left( -y_d^{(3)} - a_1(t) x_1 - a_2(t) x_2 - a_3(t) x_3 + u \right. \]
\[ \left. + c_1 (x_3 - y_d^{(2)}) + c_2 (x_2 - y_d) \right) \]

Plug in \( u \), set \( k_1(x,t) = -c_2 \)
\( k_2(x,t) = -c_1 \)

\[ \frac{d}{dt} s^2 = 2s \left[ (\gamma_1(x) - a_1(t)) x_1 + (\gamma_2(x) - a_2(t)) x_2 \right. \]
\[ \left. + (\gamma_3(x) - a_3(t)) x_3 \right] - k_3 \text{sgn} \left( s(x,t) \right) \]

\[ \therefore \text{we need} \]
\[ 2s \left[ \sum_{i=1}^{3} (\gamma_i(x) - a_i(t)) x_i - k_3 \text{sgn} (s(x,t)) \right] \leq -\varepsilon/\|s\| \]

ie. \( s > 0 \)
\[ \left[ \sum_{i=1}^{3} (\gamma_i(x) - a_i(t)) x_i - k_3 \text{sgn} (s(x,t)) \right] \leq -\varepsilon/2 \]

and if \( s < 0 \)
\[ \left[ \sum_{i=1}^{3} (\gamma_i(x) - a_i(t)) x_i - k_3 \text{sgn} (s(x,t)) \right] \geq \varepsilon/2 \]
if \( s > 0 \) \[
\begin{cases} 
\text{if } x_i > 0 \text{ let } \gamma_i < \xi_i \text{ and } k_3 > \frac{\varepsilon}{2} \\
\text{if } x_i < 0 \text{ let } \gamma_i > \beta_i \end{cases}
\]

if \( s < 0 \) \[
\begin{cases} 
\text{if } x_i > 0 \text{ let } \gamma_i > \beta_i \text{ and } k_3 > \frac{\varepsilon}{2} \\
\text{if } x_i < 0 \text{ let } \gamma_i < \xi_i 
\end{cases}
\]

\( \therefore \gamma_i(x) \) is chosen as above and \( k_3 > \frac{\varepsilon}{2} \)

Why do we reach the sliding surface in finite time?

if \( s > 0 \), \( \dot{s} \leq -\frac{\varepsilon}{2} \) so \( s(x,t) \leq -\frac{\varepsilon}{2} t + c_0 \)

where \( c_0 > 0 \)

\( \therefore s(x,t) = 0 \) for \( t \leq \frac{2c_0}{\varepsilon} \)

Similarly, if \( s < 0 \)

\( \dot{s} \geq \frac{\varepsilon}{2} \) so \( s(x,t) \geq \frac{\varepsilon}{2} t + c_0 \)

where \( c_0 < 0 \)

\( \therefore s(x,t) \to 0 \) for \( t \leq \frac{2c_0}{\varepsilon} \)

\( \therefore \) system reaches sliding surface in finite time.

On the sliding surface, \( s(x,t) = 0 \).

\( 0 = (x_3 - y_d(t)^{(2)}) + c_1 (x_2 - y_d(t)) + c_2 (x_1 - y_d(t)) \)
\[ e(t) = x_1(t) - y_2(t) \]

Then \[ \dot{e} + c_1 \dot{e} + c_2 e = 0 \]

and choose the \( c_i \) so that the characteristic polynomial has roots in the open UHP.

\[ \therefore e(t) \to 0 \text{ asymptotically.} \]

Small uncertainty vs high uncertainty?
MIMO Robust Linearization

\[
\dot{x} = f(x) + \delta f(x) + g_1(x)u_1 + \cdots + g_p(x)u_p \\
+ \delta g_1(x)u_1 + \cdots + \delta g_p(x)u_p
\]

\[
y_1 = h_1(x) \\
\vdots \\
y_p = h_p(x)
\]

(NL mimo)

Conditions for MIMO Robust Linearization

Consider the system \((NL\text{ mimo})\); suppose that the nominal system \((with \delta f, \delta g_i \equiv 0)\) has vector relative degree \(\gamma_1, \gamma_2, \ldots, \gamma_p\). Then:

If the perturbations \(\delta f, \delta g_i\) satisfy

\[
L^i f L^j_h = 0 \quad for \quad 0 \leq i \leq \gamma_j - 1
\]

\[
L^k g L^j_h = 0 \quad for \quad 0 \leq i \leq \gamma_j - 1 \quad 1 \leq k \leq p
\]

for \(j = 1 \ldots p\) then the linearization scheme for the nominal system is robust to the presence of these uncertainties \(\delta f\) and \(\delta g_k\).